Hereditarily indecomposable weakly chainable continua

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Background

Let X be a *continuum* ($\equiv$ compact connected metric space).

- X is *chainable* if for any $\varepsilon > 0$, X has a chain cover of mesh $< \varepsilon$.
  - *Chain cover*: $\langle U_1, \ldots, U_N \rangle$ where $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.
- X has *span zero* if for any continuum C and $f, g : C \to X$ with $f(C) \subseteq g(C)$ there exists $t \in C$ such that $f(t) = g(t)$.
- X is *weakly chainable* if it is a continuous image of a chainable continuum (equivalently, of the pseudo-arc).

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Theorem (Bing 1951)

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Question: Is the pseudo-arc the only (non-degenerate) hereditarily indecomposable weakly chainable continuum?

- This question is still open.
- Such a continuum is tree-like (McLean 1972, Lelek-Read 1974).

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If no... Negative answer to:

Question (Lelek 1971)

If X is a confluent image of a chainable continuum, is X chainable?

- $f : X \to Y$ is *confluent* if for each continuum $B \subseteq Y$ and each component A of $f^{-1}(B)$, $f(A) = B$.
- Every map onto a hereditarily indecomposable continuum is confluent.

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If yes. . .

Corollary

Suppose X is the image of a chainable continuum under a weakly confluent map. Then every non-degenerate hereditarily indecomposable subcontinuum of X is homeomorphic to the pseudo-arc.

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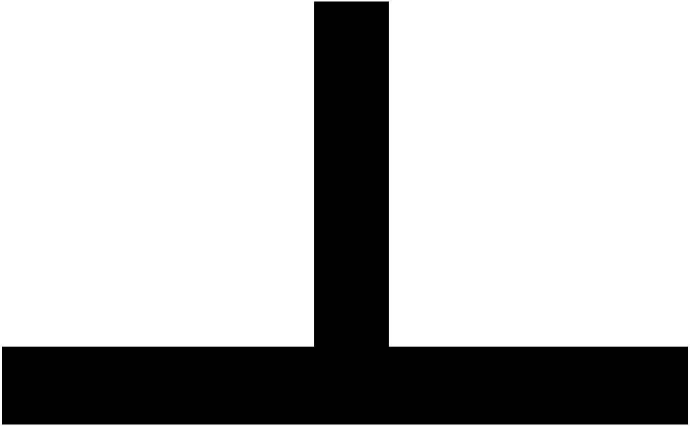
Corollary

Suppose X is a hereditarily equivalent continuum which is indecomposable. If there exists a non-constant map from the pseudo-arc to X, then X is homeomorphic to the pseudo-arc.

- X is *hereditarily equivalent* if every non-degenerate subcontinuum of X is homeomorphic to X.

Possible strategy (joint with J. Činč & L. G. Oversteegen)



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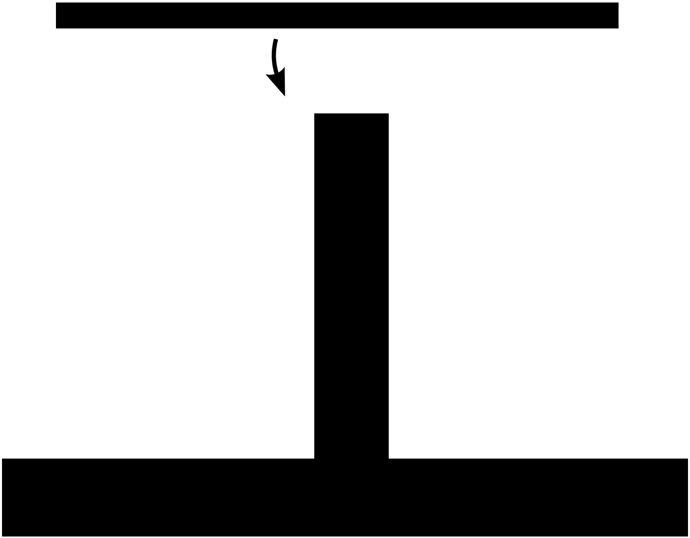

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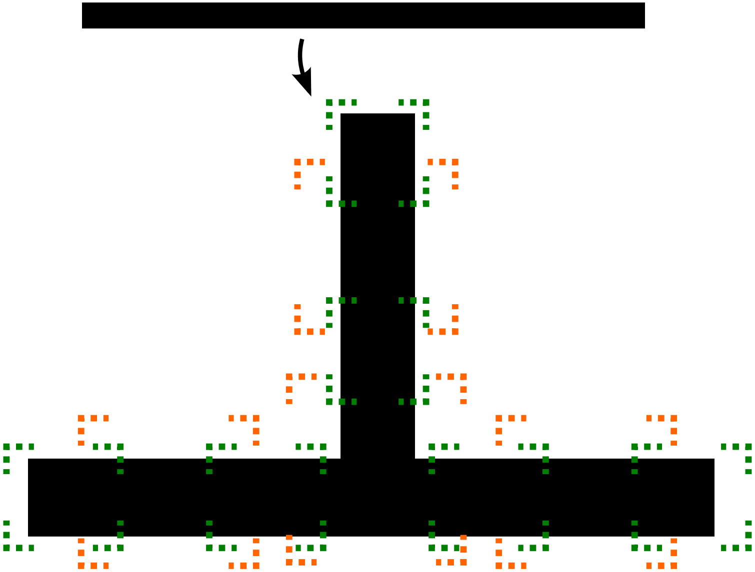

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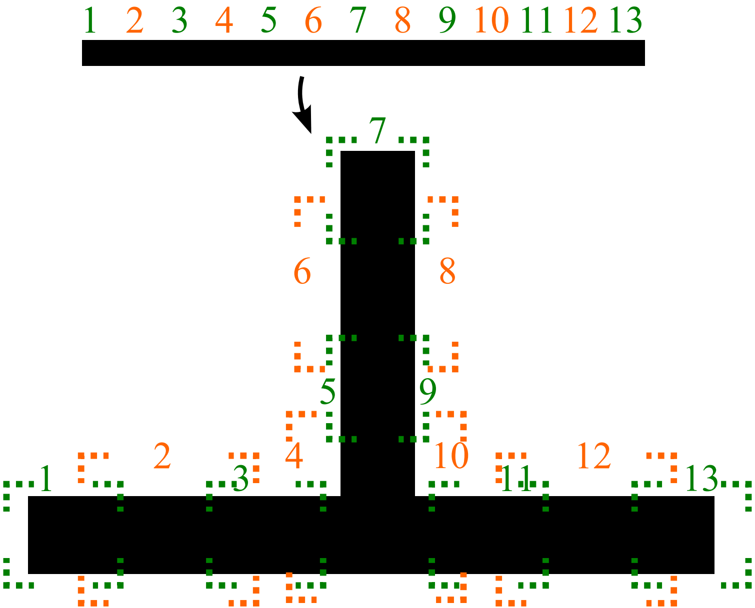

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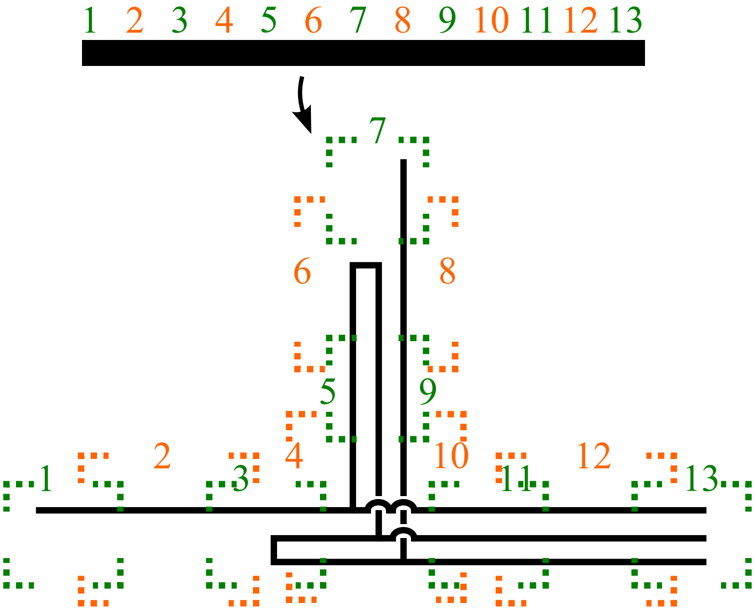

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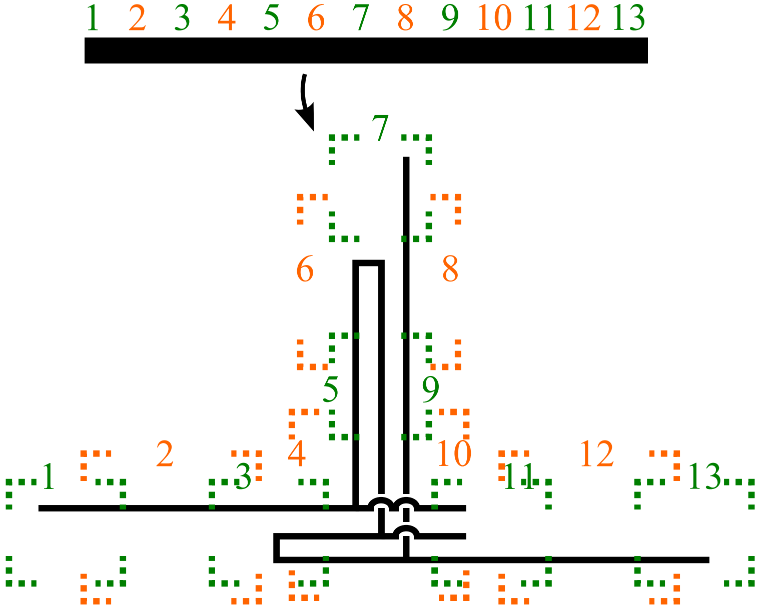

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If X is a homogeneous hereditarily indecomposable continuum, must X be weakly chainable?

- If X is homogeneous, X is hereditarily indecomposable if and only if it is tree-like (Rogers 1982, Krupski-Prajs 1990).

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