Uncountable collections of pairwise disjoint non-chainable tree-like continua in the plane

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STDC11
Definitions, Moore’s Theorem

Continuum \equiv \text{ compact connected metric space}
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Continuum $\equiv$ compact connected metric space

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- $X$ is *tree-like* if for every $\varepsilon > 0$ there is a tree $T$ and a map $f : X \to T$ whose fibres have diameters $< \varepsilon$
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Definition

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- $X$ is **arc-like**, or **chainable**, if for every $\varepsilon > 0$ there is an arc $A$ and a map $f : X \rightarrow A$ whose fibres have diameters $< \varepsilon$
- $X$ is a **triod** if there is a subcontinuum $Z \subset X$ such that $X \setminus Z$ is the union of three disjoint non-empty open sets.

Theorem (R. L. Moore, 1928)

The plane $\mathbb{R}^2$ does not contain an uncountable collection of pairwise disjoint triods.

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Homogeneous plane continua

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A space $M$ is **homogeneous** if for every $x, y \in M$ there is a homeomorphism $h : M \to M$ such that $h(x) = y$.
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The known homogeneous (non-degenerate) continua in the plane $\mathbb{R}^2$ are: the **circle** ($S^1$), **pseudo-arc**, and **circle of pseudo-arcs**.
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If this is not all of them, then by (Jones, 1955) and (Hagopian, 1976), there must be another one which is hereditarily indecomposable and tree-like.
Lemma (Hagopian, 1975)

Let $M$ be an indecomposable homogeneous continuum in the plane. Then $M$ does not contain a triod.

Proof.

Triods are decomposable, so $M$ is not a triod.

Suppose $T \subset M$ is a triod.

Since $M$ is indecomposable, it has uncountably many composants, which are pairwise disjoint; $T$ is contained in one of them.

By homogeneity, each composant of $M$ contains a copy of $T$.

This contradicts Moore's theorem.
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Non-chainable tree-like continua in the plane

Theorem (Oversteegen & Tymchatyn, 1984)

Let $M$ be an indecomposable homogeneous continuum in the plane. If every proper subcontinuum of $M$ is chainable, then $M$ is the pseudo-arc.
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Example (Ingram, 1974)

There exists an uncountable family of pairwise disjoint non-chainable tree-like continua in the plane.
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Question

Is there a non-chainable tree-like continuum $X$ such that the plane contains an uncountable collection of pairwise disjoint copies of $X$?

Example (H, 2011)

Let $X$ be the non-chainable continuum with span zero from (H, 2011). Then $X \times C$ is embeddable in the plane, where $C$ is the middle-thirds Cantor set. Moreover, if $p, q \in C$ with $|p - q| < \varepsilon$, then there is an $\varepsilon$-homeomorphism of the plane to itself taking $X \times \{p\}$ to $X \times \{q\}$.
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Open questions

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1. Is there a hereditarily indecomposable non-chainable tree-like continuum $X$ such that the plane contains an uncountable collection of pairwise disjoint copies of $X$?

2. If $X$ is a tree-like continuum and $X \times C$ embeds in the plane, must $X$ have span zero?

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