Notes on a construction of a compact metric space from a compact Hausdorff space

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All material in these notes is taken from:

◇ van der Steeg, Berd. *Models in Topology* (2003).

1 Preliminaries

1.1 Lattice bases

By a *lattice*, we mean a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound.

Example. Let $\langle X, \mathcal{T} \rangle$ be a topological space. Then $\mathcal{T}$ is a lattice (with the inclusion relation $\subseteq$), in which for $A, B \in \mathcal{T}$ we have $\sup\{A, B\} = A \cup B$ and $\inf\{A, B\} = A \cap B$.

Definition. Given a topological space $\langle X, \mathcal{T} \rangle$, a family $\mathcal{B} \subseteq \mathcal{T}$ is a *lattice base for* X provided:

(i) $\mathcal{B}$ is a base for X; and

(ii) $\mathcal{B}$ is a sublattice of 2^X (i.e. closed under finite unions and intersections).

Theorem (Wallman, 1938)**.** *There is a formula* $\alpha(x)$ *(with all quantifiers bounded by* x*) such that for any lattice* L*, we have* $\alpha(L)$ *if and only if* L *is isomorphic to a lattice base for a compact Hausdorff space.*

If $\alpha(L)$ holds, we will denote by wL some particular compact Hausdorff space for whom L is a lattice base.

1.2 Set theory and elementary submodels

Let $\phi(x_1, \ldots, x_n)$ be a formula in the language of set theory. If M is a set and $a_1, \ldots, a_n \in M$, then $M \models \phi[a_1, \ldots, a_n]$ means that ϕ holds when we treat M as the universe of sets, i.e. we replace all quantifiers in M by bound quantifiers.

For example, if $\phi(x)$ is $(\forall y)\,(\exists z)\,(x \in z \ \wedge\ y \notin z)$, then $M \models \phi[a]$ means $(\forall y \in M)\,(\exists z \in M)\,(a \in z \ \wedge\ y \notin z)$.

Given a set H, an *elementary submodel of* H is a set $M \subseteq H$ which has the property that given any formula $\phi(x_1, \ldots, x_n)$ of the language of set theory and any $a_1, \ldots, a_n \in M$, one has $H \models \phi[a_1, \ldots, a_n]$ iff $M \models \phi[a_1, \ldots, a_n]$. In this case we write $M \prec H$.

Fact (Löwenheim-Skolem)**.** Let H be a set, and let $S \subseteq H$ be countable. Then there is a countable elementary submodel M of H with $S \subseteq M$.

2 The construction

Suppose $\langle X, \mathcal{T} \rangle$ is a compact Hausdorff space, so that $\alpha(\mathcal{T})$ holds.

- Let H be a set which is large enough so that $X, \mathcal{T} \subset H$, $\mathcal{P}(X) \subset H$, $\mathcal{P}(\mathcal{P}(X)) \subset H$, and so on, to ensure that for any topological property $\Psi(x)$ we will consider below, we have $\Psi(X)$ holds if and only if $H \models \Psi(X)$.
- Let M be a countable elementary submodel of H, and define $L := \mathcal{T} \cap M$.
- From elementarity it follows that L is a sublattice of $\mathcal{T}$. We also have by elementarity $M \models \alpha(\mathcal{T})$, hence $\alpha(L)$ holds (since $\alpha(x)$ only involves quantifiers bounded by x). Thus L is (isomorphic to) a lattice base for a compact Hausdorff space wL.
- Since M is countable, L is countable, and it follows by the Urysohn metrization theorem that wL is metrizable.

The upshot is that we have a compact metric space wL and a lattice base L for wL such that, loosely speaking, L satisfies any formula that we are interested in as soon as $\mathcal{T}$ does.

3 Examples of reflected topological properties

We now consider some examples of topological properties $\Psi(x)$ for which if $\Psi(X)$ holds then $\Psi(wL)$ holds.

To avoid confusion, let us denote by 1 the top element X of $\mathcal{T}$ (and L).

§ Disconnectedness

Suppose X is disconnected. This means that there are sets $A, B \in \mathcal{T}$ such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. Written technically, we have:

$$H \models (\exists A, B \in \mathcal{T})\,(A \neq \emptyset \wedge B \neq \emptyset \wedge A \cap B = \emptyset \wedge A \cup B = 1).$$

By elementarity, M models this formula as well, hence there are $A, B \in L$ with the same property. Thus in particular there are non-empty complementary open sets in wL, so wL is disconnected.

§ Connectedness

Suppose X is connected. Then there is no pair $A, B \in \mathcal{T}$ with the properties given above. By elementarity, it follows that there is no such pair in L either.

However, a priori it may still happen that there is a non-trivial clopen subset of wL; so far we only know it cannot be in the base L. We need to argue:

Lemma 1. *Clopen subsets of a compact Hausdorff space belong to any lattice base.*

Proof. Suppose X is compact,L is a lattice base for X, and $A \subseteq X$ is clopen. Notice that A is compact. Let $\mathcal{A} \subseteq L$ be such that $\bigcup \mathcal{A} = A$. Then $\mathcal{A}$ is an open cover for A, so it has a finite subcover $\{A_1, \ldots, A_n\}$. Then $A = \bigcup_{i=1}^{n} A_i \in L$ since L is closed under finite unions. □

This completes the argument that connectedness reflects. Thus if X is a continuum, then wL is a metrizable continuum.

§ Covering dimension

Definition. Given a cover $\langle U_i \rangle_{i \in \mathcal{I}}$, a *precise refinement* of $\langle U_i \rangle_{i \in \mathcal{I}}$ is a cover $\langle V_i \rangle_{i \in \mathcal{I}}$ such that $V_i \subseteq U_i$ for every $i \in \mathcal{I}$, and $V_i \cap V_j = \emptyset$ if and only if $U_i \cap U_j = \emptyset$.

Lemma 2. *If X is a compact Hausdorff space and L is a lattice base for X, then any open cover for X has a precise refinement by members of L.*

A topological space X has covering dimension $\leq n$ iff every open cover $\langle U_i \rangle_{i=1}^{n+2}$ by $n{+}2$ sets can be shrunk to an open cover $\langle V_i \rangle_{i=1}^{n+2}$ such that $\bigcap_{i=1}^{n+2} V_i = \emptyset$. Taking precise refinements, we may assume the sets U_i and V_i belong to a given lattice base.

Suppose X has covering dimension $\leq n$. Then $H \models \forall U_1, \ldots U_{n+2} \in \mathcal{T}\, [\bigcup_{i=1}^{n+2} U_i = 1 \to \exists V_1, \ldots, V_{n+2}\, (\bigwedge_{i=1}^{n+2} V_i \subseteq U_i \wedge \bigcup_{i=1}^{n+2} = X \wedge \bigcap_{i=1}^{n+2} V_i = \emptyset)]$, hence by elementarity, M models the same sentence. It follows that wL has covering dimension $\leq n$ as well.

Likewise, if X has covering dimension $> n$, then the same holds for wL. Therefore if X has covering dimension $= n$, then wL has covering dimension $= n$.

§ Chainability

It follows from Lemma 2 that a compact Hausdorff space is chainable if and only if every open cover has a chain refinement by open sets in a (any) given lattice base. Of course it suffices to only consider covers by basic open sets.

Suppose X is chainable. Then $H \models \forall n \in \omega\, \forall U\, [(U$ is a function from n to $\mathcal{T}) \to \exists m \in \omega\, \exists V\, (V$ is a function from m to $\mathcal{T} \wedge (\forall j < m)\, (\exists i < n)\, V(j) \subseteq$

$U(i) \wedge \bigcup_{j<m} V(j) = 1 \wedge (\forall j_1, j_2 < m)\,(V(i) \cap V(j) \neq \emptyset \to |i-j| \leq 1))]$, hence by elementarity, M models the same sentence. It follows that wL is chainable.

Likewise, if X is not chainable then wL is not chainable.

§ Other properties

Some other properties that are reflected include:

- Indecomposability
- Hereditary indecomposability
- Irreducibility
- Triodicity
- (Surjective) (semi) span zero
- (Surjective) (semi) span non-zero

Remark. For the arguments for the last two items, we need to make sure the topology on $X \times X$ belongs to M as well.