Notes on a construction of a compact metric space from a compact Hausdorff space

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All material in these notes is taken from:


1 Preliminaries

1.1 Lattice bases

By a lattice, we mean a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound.

Example. Let \( \langle X, T \rangle \) be a topological space. Then \( T \) is a lattice (with the inclusion relation \( \subseteq \)), in which for \( A, B \in T \) we have \( \sup \{A, B\} = A \cup B \) and \( \inf \{A, B\} = A \cap B \).

Definition. Given a topological space \( \langle X, T \rangle \), a family \( B \subseteq T \) is a lattice base for \( X \) provided:

(i) \( B \) is a base for \( X \); and

(ii) \( B \) is a sublattice of \( 2^X \) (i.e. closed under finite unions and intersections).

Theorem (Wallman, 1938). There is a formula \( \alpha(x) \) (with all quantifiers bounded by \( x \)) such that for any lattice \( L \), we have \( \alpha(L) \) if and only if \( L \) is isomorphic to a lattice base for a compact Hausdorff space.

If \( \alpha(L) \) holds, we will denote by \( wL \) some particular compact Hausdorff space for whom \( L \) is a lattice base.

1.2 Set theory and elementary submodels

Let \( \phi(x_1, \ldots, x_n) \) be a formula in the language of set theory. If \( M \) is a set and \( a_1, \ldots, a_n \in M \), then \( M \models \phi[a_1, \ldots, a_n] \) means that \( \phi \) holds when we treat \( M \) as the universe of sets, i.e. we replace all quantifiers in \( M \) by bound quantifiers.

For example, if \( \phi(x) \) is \( \forall y (\exists z) (x \in z \land y \notin z) \), then \( M \models \phi[a] \) means \( (\forall y \in M) (\exists z \in M) (a \in z \land y \notin z) \).
Given a set $H$, an elementary submodel of $H$ is a set $M \subseteq H$ which has the property that given any formula $\phi(x_1, \ldots, x_n)$ of the language of set theory and any $a_1, \ldots, a_n \in M$, one has $H \models \phi[a_1, \ldots, a_n]$ iff $M \models \phi[a_1, \ldots, a_n]$. In this case we write $M \prec H$.

Fact (Löwenheim-Skolem). Let $H$ be a set, and let $S \subseteq H$ be countable. Then there is a countable elementary submodel $M$ of $H$ with $S \subseteq M$.

2 The construction
Suppose $\langle X, T \rangle$ is a compact Hausdorff space, so that $\alpha(T)$ holds.

- Let $H$ be a set which is large enough so that $X, T \subset H$, $\mathcal{P}(X) \subset H$, $\mathcal{P}(\mathcal{P}(X)) \subset H$, and so on, to ensure that for any topological property $\Psi(x)$ we will consider below, we have $\Psi(X)$ holds if and only if $H \models \Psi(X)$.
- Let $M$ be a countable elementary submodel of $H$, and define $L := T \cap M$.
- From elementarity it follows that $L$ is a sublattice of $T$. We also have by elementarity $M \models \alpha(T)$, hence $\alpha(L)$ holds (since $\alpha(x)$ only involves quantifiers bounded by $x$). Thus $L$ is (isomorphic to) a lattice base for a compact Hausdorff space $wL$.
- Since $M$ is countable, $L$ is countable, and it follows by the Urysohn metrization theorem that $wL$ is metrizable.

The upshot is that we have a compact metric space $wL$ and a lattice base $L$ for $wL$ such that, loosely speaking, $L$ satisfies any formula that we are interested in as soon as $T$ does.

3 Examples of reflected topological properties
We now consider some examples of topological properties $\Psi(x)$ for which if $\Psi(X)$ holds then $\Psi(wL)$ holds.

To avoid confusion, let us denote by 1 the top element $X$ of $T$ (and $L$).

§ Disconnectedness
Suppose $X$ is disconnected. This means that there are sets $A, B \in T$ such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. Written technically, we have:

$$H \models (\exists A, B \in T) (A \neq \emptyset \land B \neq \emptyset \land A \cap B = \emptyset \land A \cup B = 1).$$

By elementarity, $M$ models this formula as well, hence there are $A, B \in L$ with the same property. Thus in particular there are non-empty complementary open sets in $wL$, so $wL$ is disconnected.
§ Connectedness

Suppose $X$ is connected. Then there is no pair $A, B \in T$ with the properties given above. By elementarity, it follows that there is no such pair in $L$ either.

However, a priori it may still happen that there is a non-trivial clopen subset of $wL$; so far we only know it cannot be in the base $L$. We need to argue:

**Lemma 1.** Clopen subsets of a compact Hausdorff space belong to any lattice base.

**Proof.** Suppose $X$ is compact, $L$ is a lattice base for $X$, and $A \subseteq X$ is clopen. Notice that $A$ is compact. Let $A \subseteq L$ be such that $\bigcup A = A$. Then $A$ is an open cover for $A$, so it has a finite subcover $\{A_1, \ldots, A_n\}$. Then $A = \bigcup_{i=1}^n A_i \in L$ since $L$ is closed under finite unions. 

This completes the argument that connectedness reflects. Thus if $X$ is a continuum, then $wL$ is a metrizable continuum.

§ Covering dimension

**Definition.** Given a cover $\langle U_i \rangle_{i \in I}$, a precise refinement of $\langle U_i \rangle_{i \in I}$ is a cover $\langle V_i \rangle_{i \in I}$ such that $V_i \subseteq U_i$ for every $i \in I$, and $V_i \cap V_j = \emptyset$ if and only if $U_i \cap U_j = \emptyset$.

**Lemma 2.** If $X$ is a compact Hausdorff space and $L$ is a lattice base for $X$, then any open cover for $X$ has a precise refinement by members of $L$.

A topological space $X$ has covering dimension $\leq n$ iff every open cover $\langle U_i \rangle_{i=1}^{n+2}$ by $n+2$ sets can be shrunk to an open cover $\langle V_i \rangle_{i=1}^{n+2}$ such that $\bigcap_{i=1}^{n+2} V_i = \emptyset$. Taking precise refinements, we may assume the sets $U_i$ and $V_i$ belong to a given lattice base.

Suppose $X$ has covering dimension $\leq n$. Then $H \models \forall U_1, \ldots, U_{n+2} \in T [\bigcup_{i=1}^{n+2} U_i = 1 \rightarrow \exists V_1, \ldots, V_{n+2} (\bigwedge_{i=1}^{n+2} V_i \subseteq U_i \land \bigcup_{i=1}^{n+2} V_i = X \land \bigcap_{i=1}^{n+2} V_i = \emptyset)]$, hence by elementarity, $M$ models the same sentence. It follows that $wL$ has covering dimension $\leq n$ as well.

Likewise, if $X$ has covering dimension $> n$, then the same holds for $wL$. Therefore if $X$ has covering dimension $= n$, then $wL$ has covering dimension $= n$.

§ Chainability

It follows from Lemma 2 that a compact Hausdorff space is chainable if and only if every open cover has a chain refinement by open sets in a (any) given lattice base. Of course it suffices to only consider covers by basic open sets.

Suppose $X$ is chainable. Then $H \models \forall n \in \omega \forall U ([U$ is a function from $n$ to $T) \rightarrow \exists m \in \omega \exists V (V$ is a function from $m$ to $T \land (\forall j < m) (\exists i < n) V(j) \subseteq
\[ U(i) \land \bigcup_{j<m} V(j) = 1 \land (\forall j_1, j_2 < m) (V(i) \cap V(j) \neq \emptyset \rightarrow |i - j| \leq 1)), \text{ hence by elementarity, } M \text{ models the same sentence. It follows that } wL \text{ is chainable.} \]

Likewise, if \( X \) is not chainable then \( wL \) is not chainable.

\section*{§ Other properties}

Some other properties that are reflected include:

- Indecomposability
- Hereditary indecomposability
- Irreducibility
- Triodicity
- (Surjective) (semi) span zero
- (Surjective) (semi) span non-zero

\textit{Remark.} For the arguments for the last two items, we need to make sure the topology on \( X \times X \) belongs to \( M \) as well.