Graphs in the study of 1-dimensional continua and span

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Span and chainability

Continuum $\equiv$ compact connected metric space
Graph vs. *combinatorial graph*

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- **No** (H. 2011)
- **Yes** for hereditarily indecomposable continua (Oversteegen-H. 2016)

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2. *If $v_1 v_2 v_3 \overset{\sigma}{\hookrightarrow} a b_t c$, $v_1' v_2' v_3' \overset{\sigma}{\hookrightarrow} a b_{t'} c$, and $\varphi(v_i) = \varphi(v_i')$ for $i = 1, 2, 3$, then $|t - t'| < \frac{1}{2}$.*

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Can every continuum with span zero be embedded in $\mathbb{R}^2$?

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Theorem (Oversteegen-Tymchatyn 1984)

If X has span zero, then X is tree-like.

Simple folds

Simple fold on G: graph $F = F_1 \cup F_2 \cup F_3$ and map $\varphi : F \to G$ such that (for $G_i = \varphi(F_i)$):

1. $\partial G_2 = \partial G_1 \sqcup \partial G_3$;
2. For $i = 1, 3$ there is a neighborhood V_i of ∂G_i such that $G_i \cap V_i = G_2 \cap V_i$;
3. Each component of $G \setminus G_2$ meets only one of ∂G_1 or ∂G_2;
4. $\varphi\restriction_{F_i}$ is a homeomorphism $F_i \to G_i$; and
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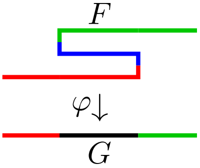

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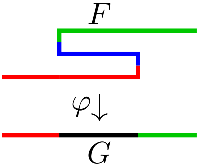

Theorem (cf. Krasinkiewicz-Minc 1977)

X is hereditarily indecomposable if and only if for any map $f : X \to G$ to a graph G, for any simple fold $\varphi : F \to G$, and for any $\varepsilon > 0$, there exists a map $g : X \to F$ such that $\varphi \circ g =_\varepsilon f$.

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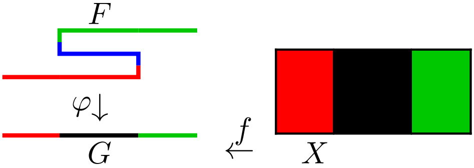

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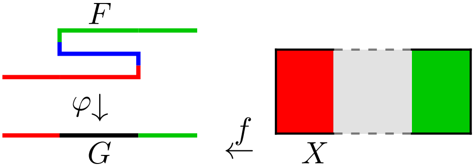

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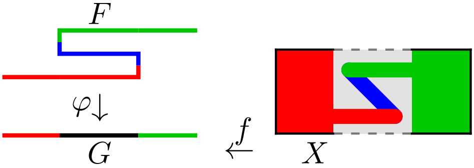

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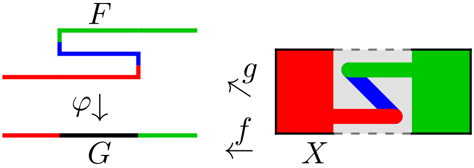

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Folding map (or *stairwell*) on G: graph $F = F_1 \cup \cdots \cup F_k$, where k is odd, and map $\varphi : F \to G$ such that (for $G_i = \varphi(F_i)$):

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