Uniformly Continuous Selections for Multi-Valued Maps

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Introduction

Theorem (Michael)

If $F: X \to Y$ is a lower semicontinuous multi-valued map of a zero-dimensional paracompact space X into a complete metric space Y and F has closed point values then F admits a continuous selection. That is $f: X \to Y$ is a continuous function such that $f(x) \in F(x)$ for each $x \in X$.

- If X is a zero-dimensional metric space it is known that one cannot always choose the selection to be uniformly continuous even if F is uniformly continuous with respect to the Hausdorff distance.

- (Y, ρ) - metric space.
- For $A \subset Y$ and $\varepsilon > 0$, $S(A, \varepsilon)$ denotes the ε-ball around A.
- Extend the definition of Hausdorff metric as follows. For closed and non-empty subsets A, B of Y let

$$H_\rho(A, B) = \inf\{\delta > 0 \colon A \subset S(B, \delta) \text{ and } B \subset S(A, \delta)\}$$

  if such δ exists. Otherwise, let $H_\rho(A, B) = \infty$.
- A multi-valued map $F \colon (X, d) \to (Y, \rho)$ between metric spaces is said to be *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $H_\rho(F(x), F(y)) < \varepsilon$.
- Note that in case the space X is compact and metric the selection $f \colon X \to Y$ in Michael's theorem is obviously uniformly continuous.

Example (Michael)

The multi-valued map G from the unit interval $[0,1]$ into the compact subsets of the space

$$\left\{\left(t, \sin\frac{1}{t}\right) \mid t \neq 0\right\} \cup \{(0,s) \mid -1 \leq s \leq 1\} \subset \mathbb{R}^2$$

defined for every $x \in X$ by the formula

$$G(x) = \begin{cases} \{(t, \sin\frac{1}{t}) \mid \frac{1}{2}x \leq t \leq x\} \text{ if } x \in (0,1], \\ \{0\} \times [-1,1] \text{ if } x = 0 \end{cases}$$

admits no continuous selection.

- Let $G' = G|_{\mathbb{Q}\cap[0,1]}$. The map G' is uniformly continuous and is defined on the zero-dimensional space. However, G' admits no uniformly continuous selection since otherwise it would extend to a continuous selection of G.

- We give one condition for which a multi-valued map F has a uniformly continuous selection.
- Recall that a metric d on a space X is an *ultrametric (or non-Archimedean)* if it satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

  for all $x, y, z \in X$.
- It is known (see [de Groot]) that a metric space X admits an ultrametric compatible with its topology if and only if $\dim X = 0$.

Result

Theorem

Let (X, d) be an ultrametric space and (Y, ρ) be a complete metric space. Then every uniformly continuous multi-valued map $F\colon (X, d) \to (Y, \rho)$ with closed point values has a uniformly continuous selection.

- Let $\{\mathcal{V}_i\}_{i=1}^{\infty}$ be the family of clopen covers of the space (X, d) such that $\mathcal{V}_i$ consists of mutually disjoint balls of radius $1/i$.
- Let F be a uniformly continuous multi-valued map with closed point values from (X, d) to a complete metric space (Y, ρ).
- For every $j \in \mathbb{N}$ there is $\delta_j > 0$ such that

$$H_\rho(F(x), F(y)) < 1/j^2$$

whenever $x, y \in X$ and $d(x, y) < \delta_j$.

- Let $\{\mathcal{W}_j\}_{j=1}^{\infty}$ be a subsequence of $\{\mathcal{V}_i\}_{i=1}^{\infty}$ such that $\operatorname{diam} V < \delta_j$ for every $V \in \mathcal{W}_j$.
- Choose $a(V, j) \in V$ for every $V \in \mathcal{W}_j$, $j \in \mathbb{N}$.
- For $j = 1$ let $b(V, 1) \in F(a(V, 1))$, $V \in \mathcal{W}_1$.
- Define a function $f_1 \colon X \to Y$ by $f_1(x) = b(V, 1)$ if $x \in V \in \mathcal{W}_1$. Then f_1 is well-defined (since there is a unique element of $\mathcal{W}_1$ which contains x) and uniformly continuous because $f_1(x) = f_1(y)$ whenever $d(x, y) < \delta_1$.
- Suppose that for $k \in \{2, \dots, n-1\}$ there are defined uniformly continuous functions $f_k \colon X \to Y$ and points $b(V, k) \in F(a(V, k))$ such that

$$\rho(b(V, k), b(U, k-1)) < 1/(k-1)^2$$

and $f_k(V) = \{b(V, k)\}$ for $V \in \mathcal{W}_k$, $U \in \mathcal{W}_{k-1}$ with $V \subset U$.

- For every $V \in \mathcal{W}_n$ there is a unique $U \in \mathcal{W}_{n-1}$ such that $V \subset U$.
- Since $\mathrm{diam} U < \delta_{n-1}$ we have $d(a(V,n), a(U,n-1)) < \delta_{n-1}$ and, hence,

$$H_\rho(F(a(V,n)), F(a(U,n-1))) < 1/(n-1)^2.$$

- There exists $b(V,n) \in F(a(V,n))$ such that

$$\rho(b(V,n), b(U,n-1)) < 1/(n-1)^2.$$

- Let $f_n \colon X \to Y$ be defined by $f_n(x) = b(V,n)$ if $x \in V \in \mathcal{W}_n$. Then f_n is uniformly continuous.
- By induction we obtain a Cauchy sequence $\{f_n\}$ of uniformly continuous functions which converges to some uniformly continuous function $f \colon X \to Y$.

- If $x \in X$ then
$$x \in \dots V_j \subset V_{j-1} \subset \cdots \subset V_1$$
for some unique sequence of $V_j \in \mathcal{W}_j$, $j \in \mathbb{N}$.
- Therefore,
$$f(x) = \lim_{j\to\infty} f_j(x) = \lim_{j\to\infty} b(V_j, j) \in Y$$
because $\{b(V_j, j)\}$ is Cauchy and Y is complete.
- Since $a(V_j, j)$ converges to x and F is uniformly continuous, $F(a(V_j, j))$ converges to $F(x)$ and, hence, $f(x) = \lim_{j\to\infty} b(V_j, j) \in F(x)$.
So $f(x) \in F(x)$ for every $x \in X$ and f is a uniformly continuous selection of F.

References

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THANK YOU