Extensions of uniformly continuous and Lipschitz functions and metrics

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Introduction

The theory of extensions of continuous functions and (pseudo)metrics has a long history. The Tietze-Urysohn theorem asserts that every continuous real-valued function on a closed subset of a metric space X admits a continuous extension to X. Hausdorff proved an analogous theorem for metrics. McShane showed that every uniformly continuous, real-valued function from a closed subset of a metric space X which admits a concave modulus function φ such that $\lim_{t \to 0} \varphi(t) = 0$ has a uniformly continuous extension to X. We show that McShane's technique may be modified to give a continuous extension operator for several classes of functions. We also use a modification of Bing's formula to construct continuous operators extending uniformly continuous metrics and ultrametics defined on the family of closed subsets of a metric (ultrametric) space.

Preliminary results

Theorem (Dugundji,1951)

Let X be a metric space and A its closed subset. Let $C^*(A)$ denote the space of all bounded real-valued continuous functions with supnorm metric. There exists a continuous, regular(of unit norm), linear extension operator $\Phi\colon C^*(A) \to C^*(X)$.

Theorem (C.Bessaga, 1993, T.Banakh, 1994, O. Pikhurko, 1994 and M. Zarichnyi, 1996)

Let (X, d) be a metric space and A a closed subset of X. There exists a continuous, regular, linear extension operator from the set of continuous (pseudo)metrics on A to the set of continuous (pseudo)metrics on X.

Preliminary results

Theorem (Stepanova, 1993)

Let (X, d) be a metric space. There exists a continuous extension operator from the space of real-valued functions whose domains are compact subsets of X to $C^*(X)$.

Theorem (H.P.Kunzi, L.Shapiro, 1997)

There exists a continuous, linear, regular extension operator for continuous, real-valued functions defined on compact subsets of a metric space.

Theorem (Tymchatyn, Zarichnyi, 2004)

Let (X, d) be a metric compactum. There exists a regular, linear, continuous with respect to the uniform topology operator extending continuous pseudometrics defined on closed subsets of X.

Notation and auxiliary results

A function $\varphi\colon [0,\infty) \to [0,\infty)$ is *concave* if φ is continuous and the set $\{(x,t) : 0 \le x \text{ and } 0 \le t \le \varphi(x)\}$ is a convex subset of $\mathbb{R} \times \mathbb{R}$. Let

$$\mathcal{J} = \{\varphi \in C([0,\infty)) : \varphi \text{ is bounded, concave, non-decr. and } \varphi(0) = 0\}.$$

(X, d) - bounded metric space.
If $A \subset X$, $f\colon A \to \mathbb{R}$ and $\varphi \in \mathcal{J}$ so that $|f(x) - f(y)| \le \varphi(d(x,y))$ for all $x, y \in A$ then we call φ a *modulus function for f*.
If ρ is a metric on A and $\varphi \in \mathcal{J}$ so that $\rho(x,y) \le \varphi(d(x,y))$ for all $x, y \in A$ then we call φ a *modulus function for ρ*.
$\exp(X)$ - the set of closed non-empty subsets of X.
$C^*_u(A)$ - the set of uniformly continuous and bounded real-valued functions on A for $A \in \exp(X)$.
We write $\mathrm{dom} f = A$ if $f \in C^*_u(A)$.

$$C^*_u = \bigcup \{ C^*_u(A) \mid A \in \exp(X)\}.$$

Each $f \in C^*_u$ is identified with its graph $\Gamma_f = \{(x, f(x)) : x \in \mathrm{dom} f\}$ which is a bounded and closed set in $X \times \mathbb{R}$.

Notation and auxiliary results

$\tilde{d}$ -the metric on $X \times \mathbb{R}$ defined by $\tilde{d}[(x,t),(x',t')] = d(x,x') + |t - t'|$.
For $f, g \in C_u^*$, $H(f,g)$ is the Hausdorff distance between Γ_f and Γ_g.
$\|f\| = \sup\{|f(x)| : x \in \mathrm{dom} f\}$ for $f \in C_u^*$.

Proposition

Let $A \in \exp(X)$ and $f : A \to \mathbb{R}$ a continuous function. Then $f \in C_u^*$ iff there is $\varphi_f \in \mathcal{J}$ which is the least modulus function for f.

Proposition

If $\{f_i\}_{i=1}^{\infty} \subset C_u^*$ with $\lim_{i\to\infty} f_i = f_0$ in C_u^* then $\varphi_{f_i} \to \varphi_{f_0}$ uniformly.

For $A \in \exp X$, $\mathcal{L}(A)$ - the set of all Lipschitz real-valued functions on A.
$\mathcal{L} = \bigcup\{\mathcal{L}(A) : A \in \exp(X)\} \subset C_u^*$. For every $f \in \mathcal{L}$ let

$$\|f\|_{\mathrm{Lip}} = \sup_{x,y \in \mathrm{dom} f, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Extension of functions

Theorem

There exists an operator $v\colon C_u^* \to C_u^*(X)$ *which satisfies the following conditions for every* $f \in C_u^*$.

1) $v(f)$ *is an extension of* f *over* X;
2) v *is regular i.e.* $\|v(f)\| = \|f\|$;
3) v *is positive homogeneous i.e.* $v(cf) = cv(f)$ *for every* $c > 0$;
4) v *is a continuous map;*

Proof.

For every $f \in C_u^*$ let φ_f be the least concave modulus function for f.
Define an operator $v\colon C_u^* \to C_u^*(X)$ by
$v(f)(x) = \min\{\inf_{y \in \mathrm{dom} f}(f(y) + \varphi_f(d(x,y))),\ \|f\|\}$ for every $x \in X$. □

Corollary

$v|_{\mathcal{L}}\colon \mathcal{L} \to C_u^*(X)$ *preserves Lipschitz functions and Lipschitz norms.*

Extension of metrics

For $A \in \exp(X)$ with $|A| \geq 2$ let $\mathcal{M}(A)$ be the set of all uniformly continuous, bounded metrics on A. Each partial metric is identified with its graph which is closed and bounded in $X \times X \times \mathbb{R}$.

$$\mathcal{M} = \bigcup\{\mathcal{M}(A) : A \in \exp(X),\ |A| \geq 2\} \subset (\exp(X \times X \times \mathbb{R}), H)$$

where H is the Hausdorff metric, generated by the l_1 metric on $X \times X \times \mathbb{R}$.
For $A \in \exp(X)$, $\mathrm{dom}\rho = A$ if $\rho \in \mathcal{M}(A)$.
For $\rho \in \mathcal{M}$ let $\|\rho\| = \sup\{\rho(x, y) : x, y \in \mathrm{dom}\rho\}$.
For $A \in \exp X$ with $|A| \geq 2$ let $\mathcal{LM}(A)$ be the set of all Lipschitz metrics on A.

$$\mathcal{LM} = \bigcup\{\mathcal{LM}(A) : A \in \exp(X), |A| \geq 2\} \subset \mathcal{M}.$$

For every $\rho \in \mathcal{LM}$ let

$$\|\rho\|_{\mathrm{Lip}} = \sup_{x,y \in \mathrm{dom}\rho, x \neq y} \frac{\rho(x, y)}{d(x, y)}.$$

Extension of metrics

Theorem

Let (X, d) be a bounded metric space. There exists an operator $u\colon \mathcal{M} \to \mathcal{M}(X)$ which has the following properties for every $\rho \in \mathcal{M}$:

1) *$u(\rho)$ is an extension of ρ over X;*
2) *u is regular that is $\|u(\rho)\| = \|\rho\|$;*
3) *u is positive homogeneous that is $u(c\rho) = cu(\rho)$ for every $c > 0$;*
4) *u is a continuous map;*
5) *$u|_{\mathcal{LM}}\colon \mathcal{LM} \to \mathcal{M}(X)$ preserves Lipschitz metrics and norms.*

Proof.

For $\rho \in \mathcal{M}$ let φ_ρ be the smallest concave modulus function of ρ Define the metric $\sigma_\rho\colon X \times X \to \mathbb{R}$ by $\sigma_\rho(x, y) = \varphi_\rho(d(x, y))$. For $x, y \in X$ let

$$u(\rho)(x, y) = \min\left\{ \inf_{a,b \in \mathrm{dom}\rho} \Big(\sigma_\rho(x, a) + \rho(a, b) + \sigma_\rho(b, y) \Big), \sigma_\rho(x, y) \right\}.$$

Extension of ultrametrics

A metric r on a set Y is called an ultrametric if $r(x,y) \leq \max\{r(x,z), r(y,z)\}$ for all $x,y,z \in Y$. Let (X,d) be a bounded ultrametric space.

For $A \in \exp X$ with $|A| \geq 2$ let $\mathcal{UM}(A)$ be the set of all uniformly continuous, bounded ultrametrics on A.

$\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp(X), |A| \geq 2\} \subset \mathcal{M}$.

Theorem

Let (X,d) be a bounded ultrametric space. There exists an operator $\alpha\colon \mathcal{UM} \to \mathcal{UM}(X)$ which has the following properties for every $\rho \in \mathcal{UM}$.

1) *$\alpha(\rho)$ is an extension of ρ over X;*
2) *α is regular that is $\|\alpha(\rho)\| = \|\rho\|$;*
3) *α is positive homogeneous that is $\alpha(c\rho) = c\alpha(\rho)$ for every $c > 0$;*
4) *α is a continuous map;*
5) *α preserves Lipschitz ultrametrics and Lipschitz norms.*

Extension of ultrametrics

Proof.

If $\rho \in \mathcal{UM}$, φ_ρ is the smallest concave modulus function for ρ and $\sigma_\rho \colon X \times X \to \mathbb{R}$ is the metric on X defined by $\sigma_\rho(x,y) = \varphi_\rho(d(x,y))$ for all $x, y \in X$. Then σ_ρ is in fact an ultrametric. Indeed

$$\sigma_\rho(x,y) = \varphi_\rho(d(x,y)) \le \varphi_\rho(\max\{d(x,z), d(z,y)\}) =$$

$$\max\{\varphi_\rho(d(x,z)), \varphi_\rho(d(y,z))\} = \max\{\sigma_\rho(x,z), \sigma_\rho(y,z)\}.$$

For $x, y \in X$ let

$$\alpha(\rho)(x,y) = \min\left\{ \inf_{a,b \in \mathrm{dom}\rho} \max\{\sigma_\rho(x,a), \rho(a,b), \sigma_\rho(b,y)\}, \sigma_\rho(x,y) \right\}.$$

□

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