A CONTINUOUS OPERATOR EXTENDING FUZZY ULTRAMETRICS

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Basic facts

Definition 1. *A binary operation* $*\colon [0,1]\times[0,1]\to[0,1]$ *is said to be a t-norm if* $*$ *is associative, commutative,* $a*1=a$ *for every* $a\in[0,1]$ *and* $a*b\le c*d$ *whenever* $a\le c$ *and* $b\le d$ *for any* $a,b,c,d\in[0,1]$.

If the map $*$ is also continuous then it is called a continuous t-norm.

Definition 2. *A triple* $(X,M,*)$ *is called a fuzzy metric space if* X *is a nonempty set,* $*$ *is a continuous t-norm and* M *is a real-valued map on* $X\times X\times(0,\infty)$ *such that for every* $x,y,z\in X$ *and* $t,s>0$ *we have*

1) $0<M(x,y,t)\le 1$;

2) $M(x,y,t)=1$ *if and only if* $x=y$;

3) $M(x,y,t)=M(y,x,t)$;

4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;

5) $M(x,y,\cdot)\colon(0,\infty)\to(0,1]$ *is continuous.*

Definition 3. *Let* $(X, M*)$ *be a fuzzy metric space. For* $x \in X$, $r \in (0,1)$ *and* $t > 0$ *the set* $B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$ *is called the open ball of radius* r *centered at* x *with respect to* t.

The family of all open balls in a fuzzy metric space forms a base of a metrizable topology.

Definition 4. *Let* $(X, M, *)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in* X *is called Cauchy if for every* $\varepsilon \in (0,1)$ *and every* $t > 0$ *there exists a positive integer* n_0 *such that* $M(x_n, x_m, t) > 1 - \varepsilon$ *whenever* $n, m > n_0$.

Definition 5. *A fuzzy metric space is called complete if every Cauchy sequence in it is convergent.*

Definition 6. *A fuzzy metric space* $(X, M, *)$ *is called stationary if* M *does not depend on* t *that is* $M(x,y,t) = M(x,y,s)$ *for every* $x, y \in X$ *and* $t, s > 0$.

Definition 7. *A map* $M\colon X \times X \times (0, \infty)$ *is called a fuzzy pseudometric if it satisfies conditions* 1), 3), 4) *and* 5) *of* Definition 2 *and additionally*

2′) $M(x,y,t) = 1$ *whenever* $x = y$ *for every* $t > 0$.

Examples of continuous t-norms: the minimum $\wedge$, the product $\cdot$, the Lukasiewicz t-norm $\mathcal{L}$ defined by $x\mathcal{L}y = \max\{0, x + y - 1\}$.

Definition 8. *A triple* $(X, M, *)$ *is called a fuzzy ultrametric space if* X *is a nonempty set,* $*$ *is the minimum* $\wedge$ *and* M *is a fuzzy set on* $X \times X \times (0, \infty)$ *satisfying conditions* 1), 2), 3) *and* 5) *of* Definition 2 *and moreover*

$4')$ $M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$ *for* $x, y, z \in X$ *and* $t > 0$.

Condition $4')$ is equivalent to the condition

$M(x, y, t) * M(y, z, t) \leq M(x, z, t)$.

We assume that the t-norm $*$ is the minimum $\wedge$.

Notation

- $(X, M, \wedge)$ – complete fuzzy ultrametric space.
- $d_M \colon X \times X \to [0, \infty)$ defined by $d_M(x, y) = 1 - \inf_{t>0} M(x, y, t)$ is a bounded ultrametric on X.
- $\exp(X)$ – the set of all closed, nonempty subsets of X with the Hausdorff metric topology generated by d_M. Then $(\exp(X), H_{d_M})$ is a bounded ultrametric space, H_{d_M} – Hausdorff metric.
- $\mathcal{F}(A)$ – the set of all fuzzy ultrametrics on A, $A \in \exp(X)$.

  $\mathcal{F}(A)$ is closed under the operation $\wedge$ applied pointwise to its elements.

- For $c \in (0,1]$ and $N \in \mathcal{F}(A)$ the map $c \odot N \colon A \times A \times (0,\infty)$ defined by $c \odot N(x,y,t) = 1 - c + cN(x,y,t)$ for $x, y \in A$ and $t > 0$ is a fuzzy ultrametric on A. We use the symbol $\odot$ to indicate the described operation applied to any real function.

- $\mathrm{dom}N = A$ if $N \in \mathcal{F}(A)$.

- $\mathcal{F} = \cup\{\mathcal{F}(A) : A \in \exp(X),\ |A| \geq 2\}$ – the family of all fuzzy ultrametrics defined on closed, non-degenerate subsets of X.

- For $N \in \mathcal{F}$ let

$$\alpha_N = \inf\{N(x,y,t) : (x,y,t) \in \mathrm{dom}N \times \mathrm{dom}N \times (0,\infty)\}.$$

  Note that $\alpha_N \in [0,1)$ for every $N \in \mathcal{F}$.

- σ – the ultrametric on $X \times \exp(X)$ defined by $\sigma((x,A),(y,B)) = \max\{d_M(x,y), H_{d_M}(A,B)\}$.

- $K = \{(x,A) \in X \times \exp(X) : x \in A\}$ is a closed subset of $X \times \exp(X)$.

- $S((x,A),r)$ – the open ball of radius $r > 0$ centered at (x,A) in the metric space $(X \times \exp(X), \sigma)$.

Extension of fuzzy ultrametrics

We consider the problem of simultaneous extension of the elements of $\mathcal{F}$ over X.

Lemma. *For a complete ultrametric space* Y *let* $\exp Y$ *be the set of its closed and bounded subsets with the Hausdorff metric. Then there exists a uniformly continuous function* $f\colon Y \times \exp Y \to Y$ *such that* $f(y, B) \in B$ *for every* $y \in Y$, $B \in \exp Y$ *and* $f(y, B) = y$ *whenever* $y \in B$.

Theorem 1. *Let* $(X, M, \wedge)$ *be a complete fuzzy ultrametric space. There exists an operator* $u\colon \mathcal{F} \to \mathcal{F}(X)$ *with the following properties for every* $N, P \in \mathcal{F}$ *and* $c \in (0, 1]$:

a) $u(N)$ *extends* N *over* X *that is* $u(N)(x, y, t) = N(x, y, t)$ *for every* $x, y \in \mathrm{dom}N$ *and* $t > 0$;

b) $u(c \odot N) = c \odot u(N)$;

c) $u(N \wedge P) = u(N) \wedge u(P)$ *whenever* $\mathrm{dom}N = \mathrm{dom}P$;

d) $\alpha_{u(N)} = \alpha_N$.

Comments on proof. (X, d_M) is complete since $(X, M, \wedge)$ is. By the above lemma there is a uniformly continuous function $f\colon X \times \exp(X) \to X$ such that $f(x, A) \in A$ for $x \in X$, $A \in \exp(X)$ and $f(x, A) = x$ if $x \in A$.

For $i \in \mathbb{N}_+$ let $\mathcal{V}_i = \{S((x, A), 1/i) : (x, A) \in X \times \exp(X)\}$.

The members of $\mathcal{V}_i$ are pairwise disjoint.

Recall that $K = \{(x, A) \in X \times \exp(X) : x \in A\}$.

For $i \in \mathbb{N}_+$ let $V_i = \bigcup\{U \in \mathcal{V}_i : U \cap K \neq \varnothing\}$. V_i is clopen in $X \times \exp(X)$ and $K \subset V_{i+1} \subset V_i$ for each $i \in \mathbb{N}_+$.

$$\mathcal{W}_i = \{V_i\} \cup \{V \in \mathcal{V}_i : V \cap K = \varnothing\}$$

is a pairwise disjoint clopen cover of $X \times \exp(X)$ and $\mathcal{W}_{i+1}$ refines $\mathcal{W}_i$.

For each $i \in \mathbb{N}_+$ define $w_i\colon \exp(X) \to \mathbb{R}^{X \times X}$ by

$$w_i(A)(x, y) = \begin{cases} \frac{1}{2} \text{ if } (x, A) \text{ and } (y, A) \text{ lie in distinct elements of } \mathcal{W}_i; \\ 1 \text{ if } (x, A) \text{ and } (y, A) \text{ lie in the same element of } \mathcal{W}_i. \end{cases}$$

Let $w\colon \exp(X) \to \mathbb{R}^{X \times X}$ be defined by the formula

$$w(A)(x, y) = \min_{i \in \mathbb{N}_+} \left\{ \frac{1}{i} \odot w_i(A)(x, y) \right\}$$

for $x, y \in X$ and $A \in \exp(X)$.

$w(A)$ is continuous with respect to x and y for every $A \in \exp(X)$.

Since $\alpha_N \in [0, 1)$ we get $(1 - \alpha_N) \in (0, 1]$ for every $N \in \mathcal{F}$.

Define a map $u\colon \mathcal{F} \to \mathcal{F}(X)$ by the formula

$$u(N)(x, y, t) = \min\{N(f(x, \mathrm{dom}N), f(y, \mathrm{dom}N), t),$$

$$(1 - \alpha_N) \odot w(\mathrm{dom}N)(x, y)\}.$$

Let $x, y \in \mathrm{dom}N$ and $t > 0$. We obtain $f(x, \mathrm{dom}N) = x$, $f(y, \mathrm{dom}N) = y$ and $w_i(\mathrm{dom}N)(x, y) = 1$ for every $i \in \mathbb{N}_+$. Then
$\frac{1}{i} \odot w_i(\mathrm{dom}N)(x, y) = 1$ for every $i \in \mathbb{N}_+$ and so $w(\mathrm{dom}N)(x, y) = 1$.
Since $(1 - \alpha_N) \odot w(\mathrm{dom}N)(x, y) = \alpha_N + (1 - \alpha_N) = 1$ we get
$u(N)(x, y, t) = \min\{N(x, y, t), 1\} = N(x, y, t)$. Therefore, u is an extension operator.

It can be shown that for every $N, P \in \mathcal{F}$ and $c \in (0, 1]$,
$u(c \odot N) = c \odot u(N)$,
$u(N \wedge P) = u(N) \wedge u(P)$ whenever $\mathrm{dom}N = \mathrm{dom}P$;
$\alpha_{u(N)} = \alpha_N$.

□

Continuity of the extension operator.

$\mathcal{FC}(A) \subset \mathcal{F}(A)$ – the set of continuous fuzzy ultrametrics on A, $A \in \exp(X)$ (every element of $\mathcal{FC}(A)$ is continuous on $X \times X \times (0, \infty)$).

$\mathcal{FC} = \cup\{\mathcal{FC}(A) \colon A \in \exp(X),\ |A| \geq 2\}$.

$\mathcal{FC} \ni P \mapsto \Gamma_P = \{(x, y, t, P(x, y, t)) : x, y \in \mathrm{dom} P,\ t \in (0, \infty)\}$

Γ_P is closed in $X \times X \times (0, \infty) \times (0, 1]$.

Define a metric ρ on $X \times X \times (0, \infty) \times (0, 1]$ by

$\rho[(a, b, t, s), (a', b', t', s')] = d_M(a, a') + d_M(b, b') + |t - t'| + |s - s'|$.

$d_M(x, y) = 1 - \inf_{t>0} M(x, y, t)$ – ultrametric on X .

H_ρ – the Hausdorff metric on $\exp(X \times X \times (0, \infty) \times (0, 1])$ generated by ρ.

H_ρ takes values in $[0, \infty]$.

$\mathcal{FC}$ – subspace of the metric space $(\exp(X \times X \times (0, \infty) \times (0, 1]), H_\rho)$.

The Hausdorff distance between any two graphs of elements from $\mathcal{FC}$ is always finite. The unboundedness of graphs occurs due to the factor $(0, \infty)$ and all graphs extend indefinitely along this axis while the other factors X and $(0, 1]$ are bounded. So $(\mathcal{FC}, H_\rho)$ is a bounded metric space.

Theorem 2. *The restriction $u|_{\mathcal{FC}} \colon \mathcal{FC} \to \mathcal{FC}(X)$ is continuous with respect to the topology on $\mathcal{FC}$ generated by the Hausdorff metric and the topology on $\mathcal{FC}(X)$ of uniform convergence on compact sets.*