UNIFORMLY CONTINUOUS SELECTIONS FOR MULTI-VALUED MAPS

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Theorem (Michael). *Let $F\colon X \to Y$ be a lower semicontinuous multi-valued map of a zero-dimensional paracompact space X into a complete metric space Y and F with closed values. Then F admits a continuous selection i.e. a continuous function $f\colon X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.*

(Y, ρ) — metric space. For $A \subset Y$ and $\varepsilon > 0$ let $S(A, \varepsilon)$ be ε-ball around A. For closed and non-empty subsets A, B of Y let

$$H_\rho(A,B) = \begin{cases} \inf\{\delta > 0 : A \subset S(B,\delta) \text{ and } B \subset S(A,\delta)\} \text{ if exists ;} \\ \infty \text{ otherwise.} \end{cases}$$

Definition 1. *A multi-valued map* $F\colon (X,d) \to (Y,\rho)$ *is uniformly continuous if* $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ *such that* $d(x,y) < \delta$ *in* X *implies* $H_\rho(F(x),F(y)) < \varepsilon$.

Example. D — the space of right continuous, real functions on $[0,1]$ with sup-norm and limit from the left at every point and with jumps possible only at rational points of $(0,1)$. The quotient map $\pi\colon D \to D/\mathrm{C}[0,1]$ admits no uniformly continuous lifting. Let $F = \pi^{-1}$. If Z is a dense countable subset of $D/\mathrm{C}[0,1]$ then $F|_Z$ has no uniformly continuous selection.

Definition 2. *A metric* d *on a space* X *is an ultrametric if* $d(x,y) \le \max\{d(x,z), d(z,y)\}$ *for all* $x,y,z \in X$.

Theorem 1. *Let* (X,d) *be an ultrametric space and* (Y,ρ) *be a complete metric space. Then every uniformly continuous multi-valued map* $F\colon (X,d) \to (Y,\rho)$ *with closed point values has a uniformly continuous selection.*

Comments on the proof. $\{\mathcal{V}_i\}_{i=1}^{\infty}$ — the family of clopen covers of the space (X, d) such that $\mathcal{V}_i$ consists of mutually disjoint balls of radius $1/i$. $\forall\ j \in \mathbb{N}\ \exists\ \delta_j > 0$ such that

$$H_\rho(F(x), F(y)) < 1/j^2$$

whenever $x, y \in X$ and $d(x, y) < \delta_j$.

$\{\mathcal{W}_j\}_{j=1}^{\infty}$ — subsequence of $\{\mathcal{V}_i\}_{i=1}^{\infty}$ such that $\mathrm{diam} V < \delta_j$ for every $V \in \mathcal{W}_j$. Choose $a(V, j) \in V$ for every $V \in \mathcal{W}_j$, $j \in \mathbb{N}$.

For $j = 1$ let $b(V, 1) \in F(a(V, 1))$, $V \in \mathcal{W}_1$. Define $f_1\colon X \to Y$ by letting $f_1(x) = b(V, 1)$ if $x \in V \in \mathcal{W}_1$. f_1 is uniformly continuous.

Inductively we define uniformly continuous functions $f_n\colon X \to Y$ and points $b(V, n) \in F(a(V, n))$ such that

$$\rho(b(V, n), b(U, n-1)) < 1/(n-1)^2$$

and $f_n(V) = \{b(V, n)\}$ for $V \in \mathcal{W}_n$, $U \in \mathcal{W}_{n-1}$ with $V \subset U$.

$\{f_n\}$ — Cauchy sequence of uniformly continuous functions.

Then $f = \lim_{n\to\infty} f_n$ is a required selection.

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