Spaces of $\sigma$-finite linear measure

I. Stasyuk, E.D. Tymchatyn*

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Abstract

Spaces of finite $n$-dimensional Hausdorff measure are an important generalization of $n$-dimensional polyhedra. Continua of finite linear measure (also called continua of finite length) were first characterized by Eilenberg in 1938. It is well-known that the property of having finite linear measure is not preserved under finite unions of closed sets. Mauldin proved that if $X$ is a compact metric space which is the union of finitely many closed sets each of which admits a $\sigma$-finite linear measure then $X$ admits a $\sigma$-finite linear measure. We answer in the strongest possible way a 1989 question (private communication) of Mauldin. We prove that if a separable metric space is a countable union of closed subspaces each of which admits finite linear measure then it admits $\sigma$-finite linear measure. In particular, it can be embedded in the 1-dimensional Nöbeling space $\nu^3_1$ so that the image has $\sigma$-finite linear measure with respect to the usual metric on $\nu^3_1$.

1 Introduction

Eilenberg and Harrold [6] asked for a characterization of continua admitting finite $n$-dimensional Hausdorff measure. They obtained a number of characterizations of continua of finite linear measure. Most useful for us they proved that a space $X$ admits a finite linear Hausdorff measure if and only if it is totally regular i.e. for each $x \in X$ and for each neighbourhood $U$ of $x$ there exist uncountably many nested neighbourhoods $\{U_\alpha\}$ of $x$ with $U_\alpha \subset U$ such that $\text{Bd}(U_\alpha) \cap \text{Bd}(U_\beta) = \emptyset$ for $\alpha \neq \beta$ and with $\text{Bd}(U_\alpha)$ finite. In particular, $X$ is hereditarily locally connected, i.e., each connected subset of $X$ is locally connected.

All spaces in this paper are separable and metric. We let $(\mathbb{R}^3, d)$ denote the Euclidean 3-space with its usual metric.

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2 Preliminaries

Definition 1. Let \((X, \rho)\) be a separable metric space and \(\alpha \geq 0\). Then the \(\alpha\)-dimensional Hausdorff measure \(H^\alpha_\rho\) on \(X\) is defined by

\[
H^\alpha_\rho(A) = \sup_{\delta > 0} \inf \left\{ \sum_{1=1}^{\infty} (\text{diam}_\rho(U_i))^\alpha \mid A \subseteq \bigcup_{i=1}^{\infty} U_i \subseteq X, \text{diam}_\rho(U_i) < \delta \text{ for every } i \in \mathbb{N} \right\}
\]

for any \(A \subseteq X\). We call \(H^1_\rho\) the linear Hausdorff measure on \((X, \rho)\).

Definition 2. The \(n\)-dimensional Nöbeling space \(\nu^{2n+1}_n\) is the subspace of the Euclidean space \(\mathbb{R}^{2n+1}\) which consists of all points with at most \(n\) rational coordinates.

The space \(\nu^{2n+1}_n\) is universal for separable metric spaces of dimension at most \(n\).

Fremlin [7, Theorem 5H] proved that if \(Y\) is a space of finite linear measure then \(Y\) embeds in \((\mathbb{R}^3, d)\) so its image has finite linear measure with respect to the metric \(d\). We shall need the following strengthening of Fremlin’s result:

Theorem 1. Let \(Y\) be a space which admits a metric \(\rho\) such that \(H^1_\rho(Y) < \infty\). Then \(Y\) embeds in a continuum \(X \subset \nu^3_1\) so \(H^1_\rho(X) < \infty\).

Proof. By [3], the space \(Y\) is totally regular. Let \(Z\) be the Freudenthal compactification of \(Y\) (see [8, page 109]). Then \(Z\) is a totally regular, metric compactum because finite separators of \(Y\) separate the distinct points of \(Z\). The components of \(Z\) form a null-family of locally connected continua. By a standard argument one can adjoin to \(Z\) a countable null sequence of arcs to obtain a totally regular metric continuum \(X\) which contains \(Z\). By [2, Theorem 3] the space \(X\) is the inverse limit of an inverse sequence \((X_n, f_n^{n+1})\) of finite connected graphs and monotone, surjective bonding maps so that each \(f_n^{n+1}: X_{n+1} \to X_n\) has at most one non-degenerate fiber.

Represent the 1-dimensional Nöbeling space \(\nu^3_1\) as \(\mathbb{R}^3 \setminus (\cup_{i=1}^{\infty} A_i)\) where each \(A_i\) is a straight line in \(\mathbb{R}^3\) with each point of \(A_i\) having at least two rational coordinates. We equip \(\nu^3_1\) with the restriction of the usual metric \(d\) from \(\mathbb{R}^3\).

We may assume that \(X\) and \(\cup_{n=1}^{\infty} X_n\) are embedded in \(\nu^3_1\) so that \(X\) is also the limit of the sequence \(\{X_n\}\) in the Hausdorff metric generated by \(d\). Indeed, suppose that \(X_1\) is embedded as a polygonal graph in \(\nu^3_1\). Let \(\varepsilon_1\) be less than half the distance from the compact set \(X_1\) to \(A_1\). Assume that \(n\) is a positive integer and that \(X_1, \ldots, X_n\) are embedded as polygonal graphs in \(\nu^3_1\) and \(\varepsilon_1 > 2\varepsilon_2 > \cdots > 2^{n-1}\varepsilon_n\) are positive numbers such that for all \(1 \leq i \leq n-1,\)

1) \(|H^1_d(X_{i+1}) - H^1_d(X_i)| < 2^{-i-1}\).
2) the Hausdorff distance from \(X_{i+1}\) to \(X_i\) is less than \(2^{-i-2}\).
3) the non-degenerate distance from \(f_{i+1}^{i+1}\) has length less than \(2^{-i-1}\).
4) \(f_{i+1}^{i+1}\) is the identity off of a sufficiently small neighbourhood of the non-degenerate element of \(f_{i+1}^{i+1}\).
5) the distance from \(X_i\) to \(A_j\) is greater than \(2\varepsilon_j\) for \(j \leq i \leq n\).
Let $\varepsilon_{n+1} > 0$ be smaller than $\frac{1}{2} \min \{\varepsilon_n, d(X_1 \cup \cdots \cup X_n, A_1 \cup \cdots \cup A_{n+1})\}$. We may take $X_{n+1}$ to be a polygonal graph in $\nu_i \nu_1^3$ so that conditions 1)-5) are satisfied for $1 \leq i \leq n$. It follows that the sequence $\{X_n\}$ converges to $X$ in the Hausdorff metric. By 5), $X \subset \nu_i$ and $H_d^1(X) = \lim_{i \to \infty} H_d^1(X_i) < \infty$ by 1) and 4).

It follows trivially from Theorem 1 that every space of finite length has a compactification of finite length in $(\nu_i, d)$.

**Definition 3.** A closed subset $A$ of a complete metric space $Y$ is called a Z-set if for each open cover $U$ of $Y$ there is a function $f: Y \to Y \setminus A$ which is $U$-close to Id$_Y$, i.e. for every $y \in Y$ there is $U \in U$ with $y, f(y) \in U$. If the map $f$ can be chosen in such a way that $f(Y) \cap A = \emptyset$ then $A$ is called a Z-set.

**Definition 4.** For a space $A$ and a complete metric space $Y$ an embedding $g: A \to Y$ is called a Z-embedding if its image is a Z-set in $Y$.

**Definition 5.** Let $Y$ and $Z$ be topological spaces and let $C(Y, Z)$ denote the set of all continuous functions from $Y$ to $Z$. For each map $f: Y \to Z$ and for each open cover $S$ of $Z$ we let $B(f, S)$ denote the set of all maps in $C(Y, Z)$ that are $S$-close to $f$. Define a collection $T$ of subsets of $C(Y, Z)$ by the rule: a subset $U \subset C(Y, Z)$ is an element of $T$ if for every $f \in U$, there exists an open cover $U$ of $Z$ such that $B(f, U) \subset U$. If $U$ and $V$ are elements of $T$ such that $B(f, U) \subset U$ and $B(f, V) \subset V$ for open covers $U$ and $V$ of $Z$, then $B(f, W) \subset U \cap V$ for any open cover $W$ which refines both $U$ and $V$. The collection $T$ is called the limitation topology on $C(Y, Z)$.

It is known that the limitation topology coincides with the topology of uniform convergence with respect to all compatible metrics on $Y$ and $Z$ (see [4, Lemma 2.1.4]).

**Definition 6.** Let $n$ be a positive integer. A Polish space $Y$ is called an absolute (neighbourhood) extensor in dimension $n$, or shortly, an $A(N)E(n)$-space, if any map $f: A \to Y$, defined on a closed subspace $A$ of a Polish space $B$ with $\dim B \leq n$, can be extended to a map of the space $B$ (respectively, of a neighbourhood of $A$ in $B$) into $Y$.

**Definition 7.** A Polish space $Y$ is called strongly $A_{w,n}$-universal if any map of any at most $n$-dimensional Polish space into $Y$ can be arbitrarily closely approximated by closed embeddings.

We will need the following result (see [4, Proposition 5.1.7]).

**Proposition 1.** Let $Y$ be an at most $n$-dimensional strongly $A_{w,n}$-universal Polish $\text{ANE}(n)$-space, and $A$ a closed subspace of an at most $n$-dimensional Polish space $B$. Then each map $f: B \to Y$, such that the restriction $f|_A$ is a Z-embedding, can be arbitrarily closely approximated by $Z$-embeddings coinciding with $f$ on $A$. In particular, the set of all $Z$-embeddings of $B$ into $Y$ is a dense $G_\delta$ subset of $C(B, Y)$.

It is known that the $n$-dimensional Nöbeling space $\nu^{2n+1}$ is a strongly $A_{w,n}$-universal, $\text{ANE}(n)$-space. The following three statements are proved in [5] as Proposition 3.6, Lemma 3.2 and Proposition 3.8, respectively.

**Proposition 2.** Let $P$ be an at most $n$-dimensional Polish space and let $C(P, \nu^{2n+1})$ denote the set of all continuous functions from $P$ into $\nu^{2n+1}$ with the limitation topology. Then the set of all $Z$-embeddings of $P$ into $\nu^{2n+1}$ is a dense $G_\delta$ subset of $C(P, \nu^{2n+1})$. 

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Proposition 3. Each compact subset of $\nu_n^{2n+1}$ is a strong $Z$-set.

Proposition 4. Each homeomorphism between compact subsets of $\nu_n^{2n+1}$ can be extended to an autohomeomorphism of $\nu_n^{2n+1}$.

Definition 8. A point $x$ of a connected space $X$ is a local cut point of $X$ if it disconnects some connected neighbourhood of $x$. The local cut point $x$ is said to be of order 2 in $X$ if it has a basis of neighbourhoods with two point boundaries.

Theorem 2. If $X$ is a connected and totally regular space then $X$ has at each point an uncountable local basis of open sets $\{U_\alpha\}$ with finite boundaries and such that each boundary point of $U_\alpha$ is a point of order 2 in $X$.

Proof. Let $Y$ be a totally regular continuum containing $X$ and constructed as in the proof of Theorem 1. Each local cut point of $Y$ is a local cut point of $X$. By [10, III, 9.2] all but at most countably many local cut points of $Y$ are of order 2 in $Y$.

Definition 9. We say that a space $Y$ admits $\sigma$-finite linear measure if there is a metric $\rho$ on $Y$ and a family $\{A_i\}_{i=1}^\infty$ of closed subsets of $Y$ so $Y = \bigcup_{i=1}^\infty A_i$ and $H_\rho^1(A_i) < \infty$ for each $i$.

Theorem 3 ([9]). Let $Y$ be a compact metric space. If $Y$ may be expressed as $\bigcup_{i=1}^\infty M_i$ where each $M_i$ is closed and admits $\sigma$-finite linear measure, then $Y$ admits $\sigma$-finite linear measure.

3 Main result

Theorem 4. Let $X = \bigcup_{i=1}^\infty X_i$ where each $X_i$ is totally regular and closed in $X$. Then the space $X$ can be embedded in $\nu^3_1$ so that the image of $X$ has $\sigma$-finite linear measure with respect to the usual metric $d$ on $\nu^3_1$.

Proof. Let $h'_1: X_1 \to \tilde{X}_1 \subset \nu^3_1$ be a compactification of $X_1$ where $\tilde{X}_1$ has finite length with respect to the metric $d$ by Theorem 1. By Proposition 1, the map $h'_1: X \to \nu^3_1$ of $X$ since $\nu^3_1$ is an $\mathcal{A}_{2,1}$ space.

Let $\mathcal{U}'_1$ be a locally finite cover of $\mathbb{R}^3 \setminus \tilde{X}_1$ by open topological 3-balls such that diam$_h(U') < \min\{1/4, d(\tilde{X}_1, U')/4\}$ for each $U' \in \mathcal{U}'_1$. We denote by $\mathcal{U}_1$ the cover of $\nu^3_1 \setminus \tilde{X}_1$ which is induced by $\mathcal{U}'_1$, i.e. $\mathcal{U}_1 = \{U' \cap \nu^3_1 \mid U' \in \mathcal{U}'_1\}$. Let $\mathcal{V}_2 = \{X_2,1, X_2,2, \ldots\}$ be a locally finite in $\mathbb{R}^3 \setminus \tilde{X}_1$ closed cover of $h_1(X_2) \setminus \tilde{X}_1$ and let $\{I_{2,1}, I_{2,2}, \ldots\}$ be finite sets of local cutpoints of order 2 in $h_1(X_2)$ such that

$$I_{2,i} \subset X_{2,i}, \ X_{2,i} \cap X_{2,j} \subset I_{2,i} \cap I_{2,j} \quad \text{for} \ i \neq j, \ \bigcup_{i=1}^\infty I_{2,i} \quad \text{is discrete in} \ \mathbb{R}^3 \setminus \tilde{X}_1$$

and such that $\mathcal{V}_2$ refines $\mathcal{U}_1$. For each $i$ let $U_{2,i} \in \mathcal{U}_1$ so $X_{2,i} \subset U_{2,i}$. For each $i$ let $T_{2,i} \subset U_{2,i}$ be a polygonal tree in $\nu^3_1$ with set of endpoints $I_{2,i}$ such that $T_{2,i} \cap T_{2,j} = I_{2,i} \cap I_{2,j}$. For each $i$ let $W_{2,i} = W_{2,i} \cap \nu^3_1$ where $W_{2,i}$ is a closed polyhedral 3-ball in $\mathbb{R}^3$ such that

$$T_{2,i} \subset W_{2,i} \subset U_{2,i}, \ T_{2,i} \cap \text{Bd}(W_{2,i}) = I_{2,i} \quad \text{and} \quad W_{2,i} \cap W_{2,j} = I_{2,i} \cap I_{2,j} \quad \text{for} \ i \neq j.$$

For each $i$ let $h_{2,i}: X_{2,i} \to \tilde{X}_{2,i} \subset \text{Int}_{\nu^3_1}(W_{2,i}) \cup I_{2,i}$ be a compactification where each $\tilde{X}_{2,i}$ has finite length with respect to the metric $d$ and $h_{2,i}|_{I_{2,i}} = \text{Id}_{I_{2,i}}$. Let $\tilde{X}_2 = \bigcup_{i=1}^\infty \tilde{X}_{2,i}$. Note
that $\tilde{X}_1 \cup \tilde{X}_2$ is compact. Let $h'_2: \tilde{X}_1 \cup h_1(X_2) \to \tilde{X}_1 \cup \tilde{X}_2$ be the embedding such that $h'_2|_{\tilde{X}_1} = \text{Id}_{\tilde{X}_1}$ and $h'_2|_{X_2,i} = h_{2,i}$ for all $i$. Note that $h'_2|_{h_1(x_2 \setminus X_1)}$ is $U_i$-close to $h_1|_{X_2 \setminus X_1}$.

Since $\tilde{X}_1 \cup \tilde{X}_2$ is compact in $\nu_3^\circ$, it is a strong $Z$-set, and so $h'_2$ can be extended to an embedding $h_2$ of $\tilde{X}_1 \cup h_1(X)$ such that $h_2|_{h_1(X) \setminus X_1}$ is $U_i$-close to $h_1|_{X \setminus X_1}$.

Let $U'_2$ be a locally finite cover of $\mathbb{R}^3 \setminus (X_1 \cup X_2)$ by open topological 3-balls such that $\text{diam}_d(U') < \min\{1/8, d(\tilde{X}_1 \setminus \tilde{X}_2, U')/8\}$ for $U' \in U'_2$. We denote by $U_2$ the cover of $\nu_3^\circ \setminus (\tilde{X}_1 \cup \tilde{X}_2)$ which is induced by $U'_2$, i.e., $U_2 = \{U' \cap \nu_3^\circ \mid U' \in U'_2\}$.

Suppose now that for $1 \leq n \leq k-1$ the covers $U_n$, the spaces $\tilde{X}_n$ and the embeddings $h_n$ are defined so that the following conditions are satisfied:

1) $\tilde{X}_1 \cup \cdots \cup \tilde{X}_n$ is compact and of $\sigma$-finite linear measure in $(\nu_3^\circ, d)$,

2) $U_n$ is a cover of $\nu_3^\circ \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_n)$ induced by a locally finite cover of $\mathbb{R}^3 \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_n)$ by open topological 3-balls such that
$$\text{diam}_d(U) < \min\{2^{-n-1}, 2^{-n-1}d(\tilde{X}_1 \cup \cdots \cup \tilde{X}_n, U)\}$$
for each $U \in U_n$,

3) $h_n: h_{n-1} \circ \cdots \circ h_1(X) \cup \tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1} \to \nu_3^\circ$ is an embedding such that
$$h_n|_{\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1}} = \text{Id}_{\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1}}$$
and
$$h_n|_{h_{n-1} \circ \cdots \circ h_1(X) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1})} \text{ is } U_n-\text{close to } h_{n-1}|_{h_{n-2} \circ \cdots \circ h_1(X) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1})}.$$

Let $\mathcal{V}_k = \{X_{k,1}, X_{k,2}, \ldots\}$ be a locally finite in $\mathbb{R}^3 \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1})$ closed cover of $h_{k-1} \circ \cdots \circ h_1(X_k) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1})$ and let $\{I_{k,1}, I_{k,2}, \ldots\}$ be finite sets of local cutpoints of order 2 in $h_{k-1} \circ \cdots \circ h_1(X_k)$ such that
$$I_{k,i} \subset X_{k,i}, \ X_{k,i} \cap X_{k,j} = I_{k,i} \cap I_{k,j} \text{ for } i \neq j,$$
and such that $\mathcal{V}_k$ refines $U_{k-1}$. For each $i$ let $U_{k,i} \subset U_{k-1}$ such that $X_{k,i} \subset U_{k,i}$ and let $T_{k,i}$ be a polygonal tree in $U_{k,i}$ with set of endpoints $I_{k,i}$ with $T_{k,i} \cap T_{k,j} = I_{k,i} \cap I_{k,j}$. For each $i$ let $W_{k,i} = W'_{k,i} \cap \nu_3^\circ$ where $W'_{k,i}$ is a closed polyhedral 3-ball in $\mathbb{R}^3$ such that
$$T_{k,i} \subset W_{k,i} \subset U_{k,i}, \ T_{k,i} \cap \text{Bd}(W_{k,i}) = I_{k,i}, \text{ and } W_{k,i} \cap W_{k,j} = I_{k,i} \cap I_{k,j} \text{ for } i \neq j.$$

For each $i$ let $h_{k,i}: X_{k,i} \to \tilde{X}_{k,i}$ be $\text{Int}_{\nu_3^\circ}(W_{k,i}) \cup I_{k,i}$ be a compactification where $X_{k,i}$ has finite length with respect to $d$ be such that $h_{k,i}|_{I_{k,i}} = \text{Id}_{I_{k,i}}$. Let $\tilde{X}_k = \bigcup_{i=1}^\infty \tilde{X}_{k,i}$ and let
$$h'_k: \tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1} \cup h_{k-1} \circ \cdots \circ h_1(X_k) \to \tilde{X}_1 \cup \cdots \cup \tilde{X}_k$$
be the compactification such that
$$h'_k|_{\tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1}} = \text{Id}_{\tilde{X}_1 \cup \cdots \cup \tilde{X}_{k-1}} \text{ and } h'_k|_{X_{k,i}} = h_{k,i} \text{ for each } i.$$
Note that
\[ h'_k|_{h_{k-1} \circ \cdots \circ h_1(X_k \setminus (X_1 \cup \cdots \cup X_{k-1}))} \]
is \(U_{k-1}\)-close to \(h_{k-1} \circ \cdots \circ h_1|_{(X_k \setminus (X_1 \cup \cdots \cup X_{k-1}))}\).

Since \(X_1 \cup \cdots \cup X_k\) is compact in \(\nu_1^k\), \(h'_k\) can be extended to an embedding \(h_k\) of \(X_1 \cup \cdots \cup X_{k-1} \cup h_{k-1} \circ \cdots \circ h_1(X)\) such that
\[ h_k|_{h_{k-1} \circ \cdots \circ h_1(X)}(X_k \setminus (X_1 \cup \cdots \cup X_{k-1})) \]
is \(U_{k-1}\)-close to \(h_{k-1} \circ \cdots \circ h_1(X_k \setminus (X_1 \cup \cdots \cup X_{k-1}))\).

Let \(U'_k\) be a locally finite cover of \(\mathbb{R}^3 \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_k)\) by open polyhedral 3-balls so \(\text{diam}_d(U) < \min\{2^{-k-1}, 2^{-k-1}d(\tilde{X}_1 \cup \cdots \cup \tilde{X}_k, U)\}\) for each \(U \in U'_k\) and let \(U_k\) be the corresponding induced cover of \(\nu_1^k\).

Then by induction \(h_k\) is defined for each positive integer \(k\). Let \(h = \lim_{k \to \infty} h_k \circ \cdots \circ h_1\).

Since the sequence \(\{h_k \circ \cdots \circ h_1\}_{k=1}^\infty\) is uniformly convergent, \(h\) is a continuous function.

Since every function \(h_k \circ \cdots \circ h_1\) is one-to-one, for each \(x \in X\) there exists a positive integer \(n\) such that \(x \in X_n\) so \(h_k \circ \cdots \circ h_1(x) = h_n \circ \cdots \circ h_1(x)\) for \(k \geq n\). It follows that \(h\) is one-to-one.

If \(x \in X \setminus (X_1 \cup \cdots \cup X_k)\) and \(h_k \circ \cdots \circ h_1(x) \in U \in U_{k-1}\) then
\[ h(x) \in \text{St}^2(U, U_{k-1}) \subset \text{St}^2(U, U_{k-1}) \subset h(X) \setminus (\tilde{X}_1 \cup \cdots \cup \tilde{X}_k) \]
as in [1, Theorem 4.2]. Hence, \(h\) is open. Thus, \(h\) is an embedding of \(X\) into \(\bigcup_{i=1}^\infty \tilde{X}_i\). The space \(\bigcup_{i=1}^\infty \tilde{X}_i\) is \(\sigma\)-compact and of \(\sigma\)-finite linear measure. \(\square\)

**Note.** Theorem 4 is sharp in the following sense. It is not true that a space of \(\sigma\)-finite linear measure embeds in a compact space of \(\sigma\)-finite linear measure. For if \(X = \mathbb{Q} \times [0, 1]\) where \(\mathbb{Q}\) is the space of rational numbers then \(X\) has \(\sigma\)-finite linear measure. It is easy to see that if \(\tilde{X}\) is a metric compactification of \(X\) then each separation of \(\tilde{X}\) between \((0, 0)\) and \((0, 1)\) contains a perfect set. However, Mauldin has shown that a space with \(\sigma\)-finite linear measure has a basis of open sets with countable boundaries.

**References**


I. Stasyuk
Department of Computer Science and Mathematics, Nipissing University,
100 College Drive, Box 5002, North Bay, ON, 51B 8L7, Canada
e-mail address ihors@nipissingu.ca, i_stasyuk@yahoo.com

E.D. Tymchatyn
Department of Mathematics and Statistics, McLean Hall, University of Saskatchewan,
106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
e-mail address tymchat@math.usask.ca