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ON OPERATORS EXTENDING UNIFORMLY DISCONNECTED METRICS

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We consider the problem of simultaneous extension of continuous uniformly disconnected metrics defined on compact subsets of a zero-dimensional metrizable compact topological space. We prove that the construction of operators extending partial ultrametrics from recent publications of E. D. Tymchatyn, M. M. Zarichnyi and the author can be applied to extending uniformly disconnected metrics.

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Рассматривается задача одновременного продолжения непрерывных равномерно несвязных метрик, определенных на компактных подмножествах нульмерного метризуемого компактного топологического пространства. Доказано, что конструкция операторов продолжения частичных ультраметрик, предложенная в статьях Э. Д. Тымчатына, М. М. Заричного и автора может быть использована для продолжения равномерно несвязных метрик.

1. Introduction. The problem of extension a single continuous metric defined on a closed subset of a metrizable topological space onto the whole space was initially considered by Felix Hausdorff in [1]. Later Hausdorff's theorem on existence of such an extension was rediscovered by many authors. The problem of existence of linear operators extending the cone of (pseudo)metrics was posed and solved for some special cases by C. Bessaga (see [3], [2]). T. O. Banakh was first to obtain the complete solution of the above problem [4] (see also [5]). A simple proof of the existence of a linear operator extending metrics was presented by M. M. Zarichnyi in [6].

In addition to theorems on extension usual (pseudo)metrics one should mention various results concerning the extension of (pseudo)metrics with special properties. It was shown in [7] that there exists an operator which simultaneously extends continuous ultrametrics defined on compact subsets of a metrizable zero-dimensional compact space. It was observed that the extension operator is continuous with respect to the uniform topology on the set of partial ultrametrics. Moreover, it preserves the maximum of two ultrametrics which have a common domain and the so-called Assouad dimension of an ultrametric space (see [10], [11] for the definition). A modification of the construction from [7] allows us to obtain an extension operator which, in addition to the properties established in [7], is homogeneous

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(see [8]). In [12] the class of uniformly disconnected metrics is introduced (see the definition below). These metrics are precisely the bi-Lipschitz images of ultrametrics (see [12]). It is easy to verify that every ultrametric is in fact a 1-uniformly disconnected metric. M. Zarichnyi asked whether the operator constructed in [8] preserves uniformly disconnected metrics. We answer this question in affirmative.

2. Preliminaries. We introduce some definitions and denotations in order to recall the construction of the extension operator from [8]. Recall that a metric ρ on a set Y is called an ultrametric if $\rho(x,y) \le \max\{\rho(x,z), \rho(y,z)\}$ for all $x,y,z \in Y$. It is known that a metrizable space admits an ultrametric compatible with its topology if and only if it is zero-dimensional (see [13]). Let X be a compact zero-dimensional metrizable space. Denote by $\exp X$ the set of all compact nonempty subsets of the space X endowed with the Vietrois topology. Let $\mathcal{UM}(A)$ stand for the set of all continuous ultrametrics defined on the set A, $A \in \exp X$. Then the set $\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp X,\ |A| \ge 2\}$ consists of all partial ultrametrics defined on compact subsets of the space X. Since the graph of each partial ultrametric is a compact subset of the space $X \times X \times \mathbb{R}$, we may consider the set $\mathcal{UM}$ as a subspace of the space $\exp(X \times X \times \mathbb{R})$. For every $\sigma \in \mathcal{UM}$ we write $\operatorname{dom}\sigma = A$ if $\sigma \in \mathcal{UM}(A)$ and let $\|\sigma\| = \max\{\sigma(x,y) : x,y \in \operatorname{dom}\sigma\}$. It is easy to see that the set $\mathcal{UM}(A)$ is closed under the operation of pointwise maximum of two ultrametrics and under multiplying by a positive constant for every $A \in \exp X$, $|A| \ge 2$. Denote by $\dim_A(Y,\rho)$ the Assouad dimension of the metric space (Y,ρ) and let C stand for the standard middle-third Cantor set in $\mathbb{R}$. In the sequel, let $K = \{(x,A) \in X \times \exp X : x \in A\}$.

Definition. A metric space (Y,ρ) is called *c-uniformly disconnected*, if there exists a constant $c > 0$ such that

$$c\rho(y_0, y_n) \le \max\{\rho(y_{j-1}, y_j) : j \in \{1,\dots,n\}\}$$

for every finite chain $y_0, y_1, \dots, y_n$ of elements of the set Y.

For every $A \in \exp X$ denote by $\mathcal{UDM}(A)$ the set of all continuous uniformly disconnected metrics on A and let $\mathcal{UDM}$ be the set of all partial uniformly disconnected metrics.

3. Auxiliary results. We are going to recall a construction of an operator extending partial ultrametrics from [8]. To this end, we will need some auxiliary facts from [7] and [8].

Proposition 1 [7]. *There exists an ultrametric d on the set C such that $\dim_A(C,d) = 0$ and $d(x,y) \le 1$ for all $x,y \in C$. Moreover d takes only values of the form $1/2^k$, $k \in \mathbb{N}$ (binary rational values).*

Proposition 2 [7]. *There exists a continuous map $f\colon X \times \exp X \to C$ such that $f(K) = \{0\}$ and the restriction $f|_{(X\times\exp X)\setminus K}$ is an embedding.*

Proposition 3 [7]. *There exists a continuous map $g\colon X \times \exp X \to X$ such that $g(x,A) \in A$ and $g(x,A) = x$ if $x \in A$ for all $x \in X$, $A \in \exp X$.*

Proposition 4 [8]. *There exists a map $w\colon \mathcal{UM} \to \mathbb{R}$ with the following properties for all $\sigma \in \mathcal{UM}$ and $A \in \exp X$ with $|A| \ge 2$.*

1) $w(\sigma) > 0$;

2) w *is continuous;*

3) the restriction $w|_{\mathcal{UM}(A)}\colon \mathcal{UM}(A) \to \mathbb{R}$ *is continuous with respect to the pointwise convergence topology on* $\mathcal{UM}(A)$.

Recall that the map w can be chosen as follows. For every $\sigma \in \mathcal{UM}$ let $w(\sigma) = \sigma(l(\operatorname{dom}\sigma), r(\operatorname{dom}\sigma))$. Here $l(\operatorname{dom}\sigma)$ and $r(\operatorname{dom}\sigma)$ denote respectively the lower and the upper bounds of the set $\operatorname{dom}\sigma \subset X \subset C \subset \mathbb{R}$.

The main result of [8] is the following theorem.

Theorem 1. *There exists a map* $v\colon \mathcal{UM} \to \mathcal{UM}(X)$, *with the following properties for all* $\sigma, \sigma' \in \mathcal{UM}$:

1) v *is continuous;*
2) *the restriction* $v|_{\mathcal{UM}(A)}\colon \mathcal{UM}(A) \to \mathcal{UM}(X)$ *is continuous with respect to the topology of pointwise convergence on* $\mathcal{UM}(A)$ *and* $\mathcal{UM}(X)$ *for every* $A \in \exp X$ *with* $|A| \geq 2$;
3) $v(\sigma)$ *is an extension of* σ *over* X;
4) $\|v(\sigma)\| = \|\sigma\|$;
5) $v(\max\{\sigma, \sigma'\}) = \max\{v(\sigma), v(\sigma')\}$ *if* $\operatorname{dom}\sigma = \operatorname{dom}\sigma'$;
6) $v(c\sigma) = cv(\sigma)$ *for every constant* $c > 0$;
7) $\dim_A(X, v(\sigma)) = \dim_A(\operatorname{dom}\sigma, \sigma)$;
8) *if* σ *takes only binary rational values then so does* $v(\sigma)$.

The extension operator $v\colon \mathcal{UM} \to \mathcal{UM}(X)$ was defined by the formula

$$v(\sigma)(x,y) = \max\{\sigma(g(x,\operatorname{dom}\sigma), g(y,\operatorname{dom}\sigma)), w(\sigma)d(f(x,\operatorname{dom}\sigma), f(y,\operatorname{dom}\sigma))\}$$

for all $\sigma \in \mathcal{UM}$ and $x, y \in X$.

4. Extending uniformly disconnected metrics. In this section we are going to prove a counterpart of Theorem 1 for uniformly disconnected metrics defined on compact subsets of the space X. First note that the construction of operators from [7] and [8] can be applied to extending continuous metrics defined on compact subsets of a zero dimensional metrizable compact space without any modification. One can easily obtain counterparts of the main results from [7] and [8] for partial metrics. We now proceed with the following theorem.

Theorem 2. *For every* $\rho \in \mathcal{UDM}$ *we have* $w(\rho) \in \mathcal{UDM}(X)$. *Moreover, if* ρ *is c-uniformly disconnected, for some* $c > 0$, *then so is* $v(\rho)$.

Proof. Let ρ be an arbitrary continuous c-uniformly disconnected metric with $\operatorname{dom}\rho = A \in \exp X$. Then the extension of ρ over X is given by the formula

$$v(\rho)(x,y) = \max\{\rho(g(x,A), g(y,A)), w(\rho)d(f(x,A), f(y,A))\}$$

for every $x, y \in X$.

We show that $v(\rho)$ is c-uniformly disconnected. Fix arbitrary finite family $\{y_0, \ldots y_n\}$ of elements of the space X. Without loss of generality we may assume that $c \leq 1$. Using the fact that d is a 1-uniformly disconnected metric we obtain

$$cv(\rho)(y_0, y_n) = \max\{c\rho(g(y_0,A), g(y_n,A)), cw(\rho)d(f(y_0,A), f(y_n,A))\} \leq$$

$$\leq \max\big\{ \max\{\rho(g(y_0,A),g(y_1,A)),\dots,\rho(g(y_{n-1},A),g(y_n,A))\},$$
$$w(\rho)\max\{d(f(y_0,A),f(y_1,A)),\dots,d(f(y_{n-1},A),f(y_n,A))\}\big\} =$$
$$= \max\big\{ \max\{\rho(g(y_0,A),g(y_1,A)),w(\rho)d(f(y_0,A),f(y_1,A))\},\dots,$$
$$\max\{\rho(g(y_{n-1},A),g(y_n,A)),w(\rho)d(f(y_{n-1},A),f(y_n,A))\} =$$
$$= \max\{v(\rho)(y_0,y_1),\dots,v(\rho)(y_{n-1},y_n)\}.$$

Therefore, the metric $v(\rho)$ is c-uniformly disconnected. □

Using the proof of Theorem 1 one can show that the extension operator $v\colon \mathcal{UDM} \to \mathcal{UDM}(X)$ has the same list of properties as for the case of partial ultrametrics.

REFERENCES

1. Hausdorff F. *Erweiterung einer Homöomorphie*, Fund. Math. **16** (1930), 353–360.
2. Bessaga C. *Functional analytic aspects of geometry. Linear extending of metrics and related problems. Progress in functional analysis* (Peñíscola, 1990), North-Holland Math. Stud. **170** (1992), North-Holland, Amsterdam, 247–257.
3. Bessaga C. *On linear operators and functors extending pseudometrics*, Fund. Math. **142** (1993), no.2, 101–122.
4. Banakh T. AE(0)*–spaces and regular operators extending (averaging) pseudometrics*, Bull. Polish Acad. Sci. Math. **42** (1994), no.3, 197–206.
5. Banakh T., Bessaga C. *On linear operators extending [pseudo]metrics*, Bull. Polish Acad. Sci. Math. **48** (2000), no.1, 35–49.
6. Zarichnyi M. *Regular linear operators extending metrics: a short proof*, Bull. Pol. Acad. Sci. Math. **44** (1996), no.3, 267–269.
7. Tymchatyn E. D., Zarichnyi M. *A note on operators extending partial ultrametrics*, Comment. Math. Univ. Carolinae **46** (2005), no.3, 515–524.
8. Стасюк І. З. *Оператори одночасного продовження часткових ультраметрик* (подано у Мат. методи і фіз.-мех. поля).
9. Stasyuk I. Z. *On a homogeneous operator extending partial ultrametrics*, Mat. Stud. **22** (2004), no.1, 73–78.
10. Assouad P. *Sur la distance de Nagata*, C. R. Acad. Sci. Paris Sér. I Math **294** (1982), no.1, 31–34.
11. Assouad P. *Plongements lipschitziens dans* $\mathbb{R}^n$, Bull. Soc. Math. France **111** (1983), 429–448.
12. David G., Semmes S. *Fractured fractals and broken dreams: Self-similar geometry through metric and measure*, Oxford Lecture Ser. Math. Appl. **7**, Oxford University Press, 1997.
13. de Groot J. *Non-archimedean metrics in topology,* Proc. Amer. Math. Soc. **7** (1956), 948–953.

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