Convex metrics on non-compact spaces

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Abstract

Bing and Moise proved that every Peano continuum admits a compatible convex metric. Bing showed also that a space with property S admits an equivalent metric which is ε-close to being convex. We answer a question of Bing by proving that for every connected and locally arc-connected metric space with property S there exists a convex metric compatible with its topology.

1 Introduction

Let (X, ρ) be a metric space. We say that ρ is *convex* if for each $x \neq y \in X$ there exists an arc $A \subset X$ with end-points x and y such that A with the restriction of the metric ρ is isometric to the interval $[0, \rho(x, y)]$ in the real line (where the real line has its usual metric). Clearly, each metric space which admits a convex metric is arc-connected and locally arc-connected. We remark that some weaker notions related to convexity were considered in [1] and [5].

Bing, [1], [3], proved that every Peano continuum (i.e., a compact, connected and locally connected metric space) admits a compatible convex metric (Moise, [8], had earlier proved a special case of the same result). He asked for an extension of his theorem to non-compact spaces. It is our purpose to prove such an extension of Bing's theorem to a quite large class of metric spaces. Our approach is a modification of that of Bing. Bing constructs in a Peano continuum X a very special decreasing sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of finite partitions of X. He then assigns weights $w_n(U)$ to the elements U of $\mathcal{U}_n$. A subcollection of $\mathcal{U}_n$ has weight equal to the sum of the weights of its elements. An approximation to the distance $\rho(x, y)$ between two points x and y of X is given by the smallest of the weights $w_n(\mathcal{C}_n)$ of chains $\mathcal{C}_n$ in $\mathcal{U}_n$ from x to y. He shows that $\limsup \bigcup \mathcal{C}_n$ contains a line segment (with respect to ρ) from x to y. (Recall here that if A_n, $n = 1, 2, \dots$, are subsets of X, then $\limsup A_n$ is the set of all points $x \in X$ each neighbourhood of which intersects infinitely many sets A_n.)

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In the non-compact case, $\limsup \bigcup \mathcal{C}_n$ need not in general contain a connected subset from x to y. Consider for instance $X = [0,1] \times [0,1] - (0,1) \times \{0\}$ in its usual metric inherited from the plane and choose $x = (0,0)$ and $y = (1,0)$. Hence, in the non-compact case, extra care needs to be taken in assigning weights to the elements of $\mathcal{U}_n$ to ensure that in the above $\limsup \bigcup \mathcal{C}_n$ is a continuuum containing x and y. We do this by defining finite connected graphs $G_1 \subset G_2 \subset \dots$ in X so that G_i is akin to the 1-skeleton of the nerve of $\mathcal{U}_i$. We then assign weights to the elements of $\mathcal{U}_i$ so that the elements of $\mathcal{U}_{i+1}$ which meet G_i are relatively light and light chains in $\mathcal{U}_{i+1}$ between distant points of X lie (except possibly near their ends) along G_i. This care in assigning weights causes formidable combinatorial problems. For this reason, we confine our attention to spaces admitting sufficient finite partitions.

2 Preliminaries

Definition 1. *A metric space (M, ρ) has property S if for each $\varepsilon > 0$ there is a finite cover of M by connected sets of diameter less than ε.*

Property S is a metric property and not a topological one. For example, the real line in its usual metric does not have property S while the open interval $(0,1)$ in its usual metric has property S. If (M, d) has property S, then it is locally connected and totally bounded, hence, separable.

It is known that every locally connected metric continuum has property S. It is also known (Filippov, [7]) that there exist locally connected, connected, separable metric spaces which do not have property S in any metric. However, there are interesting examples of non-compact spaces with property S. These include the set of points in the unit disk in the plane with at least one rational coordinate and the Nöbeling space $\mathcal{N}^{2n+1}_n$ (when considered as a pseudo-interior of the Menger cube, see [6]).

For a subset A of a space M we let $\mathrm{cl}_M(A)$, $\mathrm{bd}_M(A)$ and $\mathrm{int}_M(A)$ denote the closure, boundary and interior of A in M, respectively. We will use the denotations $\mathrm{cl}(A)$, $\mathrm{bd}(A)$ and $\mathrm{int}(A)$ respectively if it is clear from the context in what space we apply the above operations.

Let (M, d) be a metric space which is connected, locally arc-connected and has property S. Let $\mathcal{U}$ and $\mathcal{V}$ be finite coverings of M by closed sets such that $\mathcal{V}$ refines $\mathcal{U}$. For each set $U \in \mathcal{U}$ let

$$\mathcal{V}(U) = \{V \in \mathcal{V} : V \subset U\}$$
$$B(U, \mathcal{V}) = \{V \in \mathcal{V}(U) : V \cap \mathrm{bd}(U) \neq \varnothing\}$$
$$I(U, \mathcal{V}) = \mathcal{V}(U) - B(U, \mathcal{V})$$
$$\mathrm{st}(U, \mathcal{V}) = \bigcup\{V \in \mathcal{V} : U \cap V \neq \varnothing\} \text{ and } \mathrm{st}^2(U, \mathcal{V}) = \mathrm{st}(\mathrm{st}(U, \mathcal{V}), \mathcal{V}).$$

Let

$$B(\mathcal{U},\mathcal{V}) = \bigcup\{B(U,\mathcal{V}) : U \in \mathcal{U}\} \text{ and } I(\mathcal{U},\mathcal{V}) = \bigcup\{I(U,\mathcal{V}) : U \in \mathcal{U}\}.$$

Moreover, if $N \subset M$ let $\mathcal{U}|_N = \{U \cap N : U \in \mathcal{U}\}$.

Definition 2. *A labelled subcollection $\mathcal{C} = \{C_1, \dots, C_m\}$ of $\mathcal{U}$ is said to be a chain provided $C_i \cap C_j \neq \varnothing$ if and only if $|i - j| \leq 1$. It is merely a weak chain if $|i - j| \leq 1$ implies that $C_i \cap C_j \neq \varnothing$.*

Definition 3. *We shall say that a finite closed covering $\mathcal{V}$ of M is a partition of M if the following conditions are satisfied for all $U, V \in \mathcal{V}$:*

- *V and int(V) are connected and locally connected, and the first one of them is regular closed while the other is regular open,*
- *if $U \neq V$ then $U \cap V = \mathrm{bd}(U) \cap \mathrm{bd}(V)$.*

Remark. Note that the above definition of a partition differs from that used by Bing. We choose to work with closed sets while Bing used open sets ([3, page 538]). Instead of a partition being a finite collection of open disjoint connected subsets whose union is dense in the space as defined by Bing we use their closures in M instead. Taking interiors of elements of our closed partition we get a Bing's partition. It is always possible to pick these sets to be regular.

Definition 4. *Two members U and V of $\mathcal{V}$ are said to be adjacent if $U \neq V$ but $U \cap V \neq \varnothing$.*

Definition 5. *If int(V) has property S for each element V of the partition $\mathcal{V}$ of M, we call such $\mathcal{V}$ an S-partition of M. If, moreover, the mesh of $\mathcal{V}$ is less than $\varepsilon > 0$, i.e. the maximum of diameters of each $V \in \mathcal{V}$ is less than ε, then we call such $\mathcal{V}$ an ε-S-partition of M.*

Again, note that one would have to use interiors of elements of our closed partition to define an ε-S-partition in Bing's sense, see [3, page 545]).

Theorem 1 (Bing, Theorem 1 and Theorem 4, [1])**.** *For each space M with property S and each positive number ε there is an ε-S partitioning of M.*

Definition 6. *Suppose that M is a connected metric space with property S. The relative distance σ for M is defined as follows:*

$$\sigma(x, y) = \inf\{ \textit{ diam } (A) : A \subset M, A \textit{ is connected and } \{x, y\} \subset A\} \textit{ for every } x, y \in M.$$

It is known that σ preserves the topology of M and the diameter of any connected set is not increased under σ. This implies that any ε-partition of M under the metric σ is an ε-partition of M under its original metric. The completion M' of M with respect to σ is a Peano continuum that is called *the complete enclosure* of M (see [3, page 543] and [10, pages 154–158]).

Proposition 1 (Bing, Lemma 1, [1]). *If M is a connected set with property S with a relative distance metric, M' is the complete enclosure of M, and D is a connected open subset of M', then $M \cap D$ is a connected open set of M. Hence if $\mathcal{U}'$ is an ε-S partition of M' then $\mathcal{U} = \mathcal{U}'|_M$ is an ε-S partition of M.*

The following property of S-partitions will be used later to construct the connected graphs $\{G_i\}$ mentioned in the introduction.

Proposition 2. *Let $\mathcal{V}$ be an S-partition of the space M. Then for every $V \in \mathcal{V}$, bd(V) is accessible from int(V), i.e., for each $x \in \mathrm{bd}(V)$ there exists an arc I such that $x \in I \subset \{x\} \cup \mathrm{int}(V)$.*

Proof. Let V be any element of the S-partition $\mathcal{V}$ of M and fix $x \in \mathrm{bd}(V)$. There exists a decreasing null nested sequence of connected open sets W_i (in M) with property S with $W_{i+1} \subset W_i$ and $\limsup W_i = \{x\}$. For each i define x_i in W_{i+1} and arcs $A_i \subset W_i$ with endpoints x_i and x_{i+1}. Let $I = (\bigcup_{i=1}^{\infty} A_i) \cup \{x\}$. Then I is a continuum and it is locally connected except possibly at the point x since a union of finitely many locally connected continua is locally connected. The set I is in fact locally connected since a continuum cannot fail to be locally connected only on a 0-dimensional set. Hence there exists an arc $I \subset \mathrm{int}(V) \cup \{x\}$ with endpoints x_1 and x. □

To prove the main result of this paper we will use partitions of the space X with even stronger conditions, in particular, brick partitions.

Definition 7. *We shall say that a partition $\mathcal{V}$ of M is a brick partition of M if the following conditions are satisfied for all $U, V \in \mathcal{V}$:*

- *int(V) is uniformly locally connected;*
- *int($U \cup V$) is uniformly locally connected.*

We emphasize again that our partitions are closed collections of sets in M and the above definition is an obvious modification of the one used by Bing in [3, page 551].

The following proposition is well-known though we have not found a proof in the literature.

Proposition 3. *Let M be a connected space with property S and let M' be its completion in the relative distance metric. Let $\mathcal{U}'$ be a brick partition of M'. Then $\mathcal{U} = \mathcal{U}'|_M = \{U' \cap M : U' \in \mathcal{U}'\}$ is a brick partition of M.*

Proof. Let $U' \in \mathcal{U}'$. By Definition 7, $\mathrm{int}_{M'}(U')$ is dense in U' and uniformly locally connected. Since $\mathrm{int}_{M'}(U') \cap M$ is open in M by Proposition 1, $\mathrm{int}_{M'}(U') \cap M \subset \mathrm{int}_M(U' \cap M)$. If $x \in M \setminus \mathrm{int}_{M'}(U') = M \setminus (\mathrm{int}_{M'}(U') \cap M)$ then there is a sequence $\{x_i\}$ in $M' \setminus U'$ converging to x since U' is closed in M'. Since M is dense in M' there is a sequence $\{y_i\}$ in $M \setminus U'$ converging to x. So $x \notin \text{int}_M(U' \cap M)$. We have proved $\text{int}_{M'}(U') \cap M = \text{int}_M(U' \cap M)$.

By Proposition 1, $\text{int}_M(U' \cap M)$ is connected and open in M. We claim that $\text{int}_M(U' \cap M)$ is uniformly locally connected. By the contrary, suppose that there exists $\varepsilon > 0$ and sequences $\{x_i\}$ and $\{y_i\}$ in $\text{int}_M(U' \cap M)$ with the distance between x_i and y_i less than $1/i$ but no connected set in $\text{int}_M(U' \cap M)$ containing x_i and y_i has diameter less than ε. Since M' is compact we may suppose there exists $z \in U'$ such that $\lim x_i = z = \lim y_i$ as $i \to \infty$. Since $\text{int}_{M'}(U')$ is uniformly locally connected there exist connected sets $V_i \subset \text{int}_{M'}(U')$ such that $x_i, y_i \in V_i$ and $\lim \text{diam} V_i = 0$. Since $\text{int}_{M'}(U')$ is open and locally connected the sets V_i can be chosen to be open. However, by Proposition 1, $V_i \cap M$ is a connected open set in $\text{int}_M(U' \cap M)$ containing x_i and y_i for each i. A contradiction. So $\text{int}_M(U' \cap M)$ is uniformly locally connected.

Finally let U' and U'' be two distinct elements of the brick partition $\mathcal{U}'$ with $U' \cap U'' \neq \varnothing$. Then $\text{int}_{M'}(U' \cup U'')$ is connected and uniformly locally connected. By Proposition 1 $\text{int}_M((U' \cup U'') \cap M)$ is connected and by the argument of the previous paragraph it is uniformly locally connected. □

Theorem 2 (Bing, Theorem 8 from [2] and Theorem 10 from [3])**.** *Each Peano continuum has a sequence $\{\mathcal{U}_i\}$ of brick partitionings such that $\mathcal{U}_i$ has mesh less that $1/i$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$.*

Definition 8. *We shall say that $\mathcal{V}$ is a core refinement of the partition $\mathcal{U}$ of M if $\mathcal{V}$ is also a partition of M which refines $\mathcal{U}$ and the following conditions are satisfied for all $U \in \mathcal{U}$:*

- *$I(U, \mathcal{V})$ is a non-empty coherent collection (i.e., $\bigcup I(U, \mathcal{V})$ is connected),*
- *each member of $B(U, \mathcal{V})$ meets at least one member of $I(U, \mathcal{V})$*

(see [3, page 545]).

It follows readily from [4, Theorem 3] that in Theorem 2 $\mathcal{U}_{i+1}$ may be taken to be a core refinement of $\mathcal{U}_i$.

Definition 9. *Let $\mathcal{U}$ be a partition of the connected, locally connected, metric space M and let G be a closed connected subset of M such that $\mathcal{U}|_G = \{U \cap G \mid U \in \mathcal{U}\}$ is a partition of G. Then $\mathcal{U}$ is called a simultaneous partition of M and G.*

Let $\mathcal{V}$ be a core refinement of $\mathcal{U}$ such that $\mathcal{V}|_G = \{V \cap G \mid V \in \mathcal{V}\}$ is a core refinement of $\mathcal{U}|_G$. Then we say that $\mathcal{V}$ is a simultaneous core refinement of $\mathcal{U}$ with respect to M and G.

Thomas [9] proved existence of simultaneous core partitions for compact sets:

Theorem 3 (Thomas, Theorem 8, [9])**.** *Let M be a Peano continuum and N a Peano subcontinuum of M. Then there exists a sequence of simultaneous core partitionings $\mathcal{U}_i$ of M and N such that $\mathcal{U}_i$ has mesh less than $1/i$ and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$.*

We are able to generalize the above result for the case of brick partitions if N is a graph.

Definition 10. *Let M be a Peano continuum and let N be a Peano continuum contained in M. Let $\mathcal{U}$ be a brick partition of M. Suppose that the following conditions are satisfied:*

- *$\mathcal{U}|_N$ is a brick partition of N*
- *if U and V are adjacent elements of $\mathcal{U}$ with $U \cap V \cap N \neq \varnothing$ then*

$$U \cap V \cap \operatorname{int}_N((U \cup V) \cap N) \cap \operatorname{int}(U \cup V) \neq \varnothing.$$

. Then $\mathcal{U}$ is called a simultaneous brick partition of M and N.

Recall a method of forming sets with property S used in [3, page 540].

Definition 11 (Gradual growing process)**.** *A set H is said to ε-grow into a set H' in a space M if M contains H', each point of H' belongs to a connected subset of H' of diameter less than ε that intersects H, and for some positive number δ, H' contains all connected subsets of M of diameter less than δ that intersect H.*

The key to constructing brick partitions of a Peano continuum is contained in [3, Lemma for Theorem 8, page 549]. Using Theorem 3 we tweak Bing's Lemma and [3, Theorems 8, 9] in order to prove existence of simultaneous brick partitionings for two Peano continua M and N with $N \subset M$.

Lemma 0. *Let M and N be Peano continua with $N \subset M$. Let C_1 and C_2 be disjoint closed subsets of M such that $M \setminus C_i$ and $N \setminus C_i$ have property S and $N \cap C_i \neq \varnothing$ for $i = 1, 2$. Suppose that $\varepsilon > 0$ is such that C_j can ε-grow into $M \setminus C_i$ and $N \cap C_j$ can ε-grow into $N \setminus C_i$ for $i \neq j$ and $i, j = 1, 2$. Then for each $\delta > 0$ M contains disjoint open sets D_1 and D_2 satisfying the following conditions for $i, j = 1, 2$ and $i \neq j$:*

(1) C_i can δ-grow into D_i and $\varepsilon + \delta$-grow into $M \setminus D_j$.

(2) $C_i \cap N$ can δ-grow into $D_i \cap N$ and $\varepsilon + \delta$-grow into $N \setminus D_j$.

(3) $M \setminus (D_1 \cup D_2)$ can δ-grow into $M \setminus (C_1 \cup C_2)$.

(4) $N \setminus (D_1 \cup D_2)$ can δ-grow into $N \setminus (C_1 \cup C_2)$.

(5) D_i, $M \setminus D_i$, $D_i \cap N$ and $N \setminus D_i$ have property S.

Proof. The proof is very similar to the one in Bing's Lemma in [3, page 549]. Using Theorem 3 suppose that $\mathcal{U}$ is a γ-S simultaneous partitioning of M and N where γ is less than both δ and the one third distance between C_1 and C_2. Let A_i (respectively, B_i) be finite collections of arcs in $N \setminus C_j$ (respectively, in $M \setminus C_j$), $i \neq j$, $i = 1, 2$ of diameter less than ε, each intersecting C_i and each element of $\mathcal{U}|_N$ (respectively, $\mathcal{U}$) intersects an arc of A_1

(respectively, B_1) and an arc of A_2 (respectively, B_2). From the hypothesis $M \setminus (C_1 \cup C_2)$ and $N \setminus (C_1 \cup C_2)$ have property S and so they are partitionable. By Theorem 3 let $\mathcal{U}'$ be a γ'-S simultaneous partitioning of $M \setminus (C_1 \cup C_2)$ and $N \setminus (C_1 \cup C_2)$ which refines $\mathcal{U}$ and γ' is less than the minimum distance between C_i and any arc of $B_j \cup A_j$, $i \neq j$, $i = 1, 2$.

Let R_0' be the union of all elements $U \in \mathcal{U}'$ such that either U lies in an element of $\mathcal{U}$ not adjacent to $C_1 \cup C_2$ or U lies in an element of $\mathcal{U}$ adjacent to C_i and U intersects an arc in $B_j \cup A_j$ $(i \neq j)$. Set R_0^* to be the union of R_0' together with all elements $U' \in \mathcal{U}'$ such that $U' \subset V \in \mathcal{U}$ where V is adjacent to C_i but R_0' separates U' from C_i in $V \cup C_i$. Let R_0 be R_0^* together with all elements $U' \in \mathcal{U}'$ such that $U' \subset V \in \mathcal{U}$ where V is adjacent to C_i but $R_0^* \cap N$ separates $U' \cap N$ from $C_i \cap N$ in $(V \cap N) \cup (C_i \cap N)$. Suppose R_n is defined. If for each $V \in \mathcal{U}$ with V adjacent to C_i there is no element $U \in \mathcal{U}'$ with $U \subset V$ separated from $C_i \cap V$ in $V \cup C_i$ by R_n and if $U \cap N \neq \varnothing$ is not separated by R_n from $C_i \cap N$ in $(V \cap N) \cup (C_i \cap N)$ then let $R = R_n$. If not repeat the same process with R_n as with R_0' to obtain R_{n+1}. Since $\mathcal{U}'$ is finite this process terminates after finitely many steps.

We define D_i to be the union of all elements of $\mathcal{U}'$ which do not lie in R but which lie in an element of $\mathcal{U}$ adjacent to C_i. The rest of the proof is as in Bing's Lemma in [3, page 549]. □

From Lemma 0 we obtain by Bing's argument the following distillation of [3, Theorems 8 and 9].

Theorem 4. *Let $N \subset M$ be Peano continua and let H and K be disjoint closed subsets of M. Then there is a continuous function $T \colon M \to [0, 1]$ such that*

1) *$T(H) = 0$ and $T(K) = 1$,*

2) *for each connected and non-degenerate set $W \subset [0, 1]$, $T^{-1}(W)$ and $T^{-1}(W) \cap N$ have property S and*

3) *for all but at most countably many $x \in [0, 1]$ the sets $T^{-1}([0, x))$, $T^{-1}([0, x)) \cap N$, $T^{-1}((x, 1])$ and $T^{-1}((x, 1]) \cap N$ are uniformly locally connected.*

Note that on each stage the conditions of Lemma 0 apply simultaneously to M and N.

Theorem 5. *Let $N \subset M$ be Peano continua. Then there is a sequence $\{\mathcal{U}_i\}$ of simultaneous brick partitions of M and N such that $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$ and mesh $\mathcal{U}_i$ is less than $1/i$ for each i.*

Proof. The proof is the proof of Bing in [2, Theorem 8] using our Theorem 4 in place of Theorems 3, 4 and 5 in [2] and noting that at each stage the results in Bing apply simultaneously to both M and N. □

We shall need the following extension of Theorem 3 to brick partitions in case the subcontinuum N is a graph.

Theorem 6. *Let M be a Peano continuum, N a compact connected graph contained in M and $\mathcal{U}$ a simultaneous brick partition of M and N such that no vertex of N lies in the boundary of any element of $\mathcal{U}$. Then for each $\varepsilon > 0$ there is a brick ε-partition $\mathcal{V}$ which is a simultaneous core refinement of $\mathcal{U}$ with respect to M and N.*

Proof. The proof is based on that of Theorem 3 in [4]. Let $\mathcal{U}'$ be a simultaneous brick partition of M and N of mesh less than ε and which refines $\mathcal{U}$. By condition 3) in Theorem 4 we may suppose that no vertex of N lies on the boundary of any element of $\mathcal{U}'$. We may also suppose that $N \cap V$ is either empty or a tree for each $V \in \mathcal{U}'$.

Let T be a connected graph in M with $N \subset T$ such that

1) T meets each $V \in \mathcal{U}'$ in a tree.
2) If V and V' are distinct elements of $\mathcal{U}'$ such that $V \cap V' \neq \varnothing$ then there is an arc in T which crosses from $\mathrm{int}(V)$ into $\mathrm{int}(V')$ at an interior point of $V \cup V'$ and that point is the unique point of this arc which is on $\mathrm{bd}(V) \cup \mathrm{bd}(V')$.
3) for each $V \in \mathcal{U}'$ $T \cap V$ is a tree with only its endpoints on $\mathrm{bd}(V)$.

For each $U \in \mathcal{U}$ let T_U be a compact connected graph in $T \cap \mathrm{int}(U)$ so that $(T \cap U) \setminus T_U$ consists of half open arcs $A_{U,1}, \ldots, A_{U,m_U}$ with pairwise disjoint closures and such that each $\mathrm{cl}(A_{U,i})$ has just the endpoint of $A_{U,i}$ on $\mathrm{bd}(U)$ and $\mathrm{cl}(A_{U,i}) \setminus \mathrm{bd}(U) \subset \mathrm{int}(V)$ for some $V \in \mathcal{U}'(U)$. Note that $V \cap T_U$ is connected for each $V \in \mathcal{U}'(U)$ by 3). Let

$$0 < \delta < \frac{1}{2} \min\Big(\big\{d(T_U, \mathrm{bd}(U)) \colon U \in \mathcal{U}\big\} \cup \big\{\mathrm{diam}(A_{U,i}) \colon U \in \mathcal{U},\ i \in \{1, \ldots, m_U\}\big\} \cup \\ \cup \big\{d(A_{U,i}, A_{U,j}) \colon U \in \mathcal{U},\ i \neq j,\ i, j \in \{1, \ldots, m_U\}\big\}\Big).$$

Let $\mathcal{U}''$ be a simultaneous brick partition of M and T such that $\mathcal{U}''$ refines $\mathcal{U}'$ and has mesh less than δ. For each $U \in \mathcal{U}$ let U_c be the largest connected set in $\mathrm{int}(U)$ which contains T_U and consists of elements of $\mathcal{U}''$.

Let $V \in I(U,\mathcal{U}')$. Then $V \subset U_c$ since V is in the connected union of members of $I(U,\mathcal{U}'')$ and $V \cap T_U \neq \varnothing$. Let $\mathcal{U}''(V) \subset I(U,\mathcal{V})$.

Let $V \in B(U,\mathcal{U}')$ and let V_c be the component of $U_c \cap V$ which meets T_U. Note that if $A_{U,i} \subset V$ then $\mathrm{st}(A_{U,i},\mathcal{U}''(V)) \subset V$. Also if $i \neq j$ then $\mathrm{st}(A_{U,i},\mathcal{U}''(V)) \cap \mathrm{st}(A_{U,j},\mathcal{U}''(V)) = \varnothing$. Let

$$K_V = V \setminus \mathrm{int}_V\Big(V_c \cup \bigcup \{\mathrm{st}(A_{U,i},\mathcal{U}''(V)) \mid A_{U,i} \subset V\}\Big).$$

Note that if C is a component of K_V and $\mathrm{cl}(C) \cap V_c \neq \varnothing$ then by definition of U_c and V_c, $\mathrm{cl}(C) \cap \mathrm{bd}(U) \neq \varnothing$. In this case we let $\mathrm{cl}(C) \in B(U,\mathcal{V})$.

If $A_{U,i} \subset V$ let

$$\mathrm{st}(A_{U,i},\mathcal{U}''(V)) \cup \bigcup \big\{\mathrm{cl}(C) \mid C \text{ is a component of } K_V, C \cap V_c = \varnothing, \\ i \text{ is smallest so that } A_{U,i} \subset V \text{ and } C \cap \mathrm{st}(A_{U,i},\mathcal{U}''(V)) \neq \varnothing\big\} \in B(U,\mathcal{V}).$$

We have defined all of the elements of $\mathcal{V}$. Then $\mathcal{V}$ so defined is the required simultaneous core brick ε-partition of M and N which refines $\mathcal{U}$.

□

3 Setup for the main result

The following theorem is the main result in this paper.

Theorem 7. *Every connected and locally arc connected metric space with property S admits an equivalent convex metric.*

Setup for the proof: Let X be a connected and locally arc connected metric space with property S and let X' be its complete enclosure. We now define a sequence of brick partitions for X' (and X) and weights for the elements of these partitions which we will use to define a convex metric on X.

By Theorem 2 there exists a brick partition $\mathcal{U}'$ of X' of mesh less than 1. By Proposition 3, the collection

$$\mathcal{U}_1 = \{\mathrm{cl}_X(X \cap \mathrm{int}(U')) : U' \in \mathcal{U}'_1\} = \{X \cap \mathcal{U}' : U' \in \mathcal{U}'_1\}$$

is a brick partition of the space X of mesh less than 1. We may suppose that $|\mathcal{U}_1| \geq 4$.

For all $U, V \in \mathcal{U}_1$ which are adjacent let $x_{U,V}$ be a point of $\mathrm{int}_X(U \cap V)$. By Proposition 2, for each $V \in \mathcal{U}_1$ let G_V be a finite, connected and simply connected graph in V such that $G_V \cap \mathrm{bd}(V) = \{x_{U,V} : U \in \mathcal{U}_1 \text{ is adjacent to } V\}$ and $G_V \cap \mathrm{int(V)}$ is connected.

Let $G_1 = \bigcup\{G_V : V \in \mathcal{U}_1\}$. Then G_1 is a finite connected graph. We shall place a metric on X so that the shortest paths between two disjoint elements of $\mathcal{U}_1$ are in G_1. We are going to define a core refinement $\mathcal{U}_2$ of the partition $\mathcal{U}_1$ with certain conditions. In order to do this we use Proposition 3 and Theorem 6.

Again we consider X', the complete enclosure of the space X and its partition $\mathcal{U}'_1$. By Theorem 6 there exists a simultaneous brick partition $\mathcal{U}'_2$ of X' and G_1. Note that $G_1 \cap V'$ is either empty or a tree for each $V' \in \mathcal{U}'_2$. Using Theorem 6 we can assume that $\mathcal{U}'_2$ is in fact a core refinement of $\mathcal{U}'_1$. By Proposition 3 the collection $\mathcal{U}_2 = \{\mathrm{cl}_X(X \cap \mathrm{int}(U')) : U' \in \mathcal{U}'_2\} = \{X \cap U' : U' \in \mathcal{U}'_2\}$ is a brick partition of X and a simultaneous core refinement of $\mathcal{U}_1$ with respect to X and G_1. We are able to extend G_1 to a graph G_2 such that the brick partition $\mathcal{U}_2$ and the graph G_2 satisfy the following conditions:

- $\mathrm{mesh}\,\mathcal{U}_2 < 2^{-1}$,
- $\mathrm{int}(U)$ has property S for each $U \in \mathcal{U}_2$,
- $\mathcal{U}_2$ is a core refinement of $\mathcal{U}_1$,
- any chain in $\mathcal{U}_2$ between two disjoint links of $\mathcal{U}_1$ has at least 16 links,

- if $W \in \mathcal{U}_2$ then $G_2 \cap \mathrm{int}(W)$ is connected, $G_2 \cap W$ is a tree and $G_2 \cap \mathrm{bd}(W)$ is finite,
- if U and W are adjacent elements of $\mathcal{U}_2$ then $U \cap W \cap G_2 \neq \varnothing$.

The fourth condition from the list above is true due to the fact that the distance between any two disjoint elements of a brick partition is positive. So if there is a chain between two disjoint bricks that has less than 16 elements we can always subdivide further one of them to get a sufficiently large number of links.

Inductively, we construct a family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of partitions of X, and a sequence $\{G_i\}_{i=1}^{\infty}$ of finite graphs in X such that

(a) $\mathrm{mesh}\,\mathcal{U}_i < 2^{-i}$,

(b) $\mathrm{int}(U)$ has property S for each $U \in \mathcal{U}_i$,

(c) $\mathcal{U}_{i+1}$ is a core refinement of $\mathcal{U}_i$,

(d) each chain in $\mathcal{U}_{i+1}$ between two disjoint elements of $\mathcal{U}_i$ contains at least 16 links,

(e) if $W \in \mathcal{U}_{i+1}$ then $G_{i+1} \cap \mathrm{int}(W)$ is connected, $G_{i+1} \cap W$ is a tree and $G_{i+1} \cap \mathrm{bd}(W)$ is finite,

(f) $G_i \subset G_{i+1}$,

g) if $U, V \in \mathcal{U}_{i+1}$ are two adjacent elements then $U \cap V \cap G_{i+1} \neq \varnothing$.

Condition (d) will play a significant role in further combinatorial considerations.

We shall construct a metric on X so that the shortest paths between two disjoint elements of $\mathcal{U}_{i+1}$ lie in G_{i+1}.

For each i let $m_i = |\mathcal{U}_i|$ denote the number of members of $\mathcal{U}_i$.

Definition 12. *For each $U \in \mathcal{U}_1$ let $w_1(U) = 1$. For each connected subset K of X such that $K = \bigcup \mathcal{V}$ for some non-void subcollection $\mathcal{V}$ of $\mathcal{U}_1$, we say that K is a set which has 1^{st}-weight and we set $w_1(K) = \sum_{V \in \mathcal{V}} w_1(V)$.*

Definition 13. *For $U \in \mathcal{U}_2$ define the position of U to be an integer $P(U) \in \{0, 1, 2, 3\}$ such that if $U \subset U' \in \mathcal{U}_1$ then*

$$P(U) = \begin{cases} 0 \text{ if } U \in B(U', \mathcal{U}_2) \text{ and } \mathrm{int}(U) \cap G_1 \neq \varnothing \\ 1 \text{ if } U \in B(U', \mathcal{U}_2) \text{ and } \mathrm{int}(U) \cap G_1 = \varnothing \\ 2 \text{ if } U \in I(U', \mathcal{U}_2) \text{ and } \mathrm{int}(U) \cap G_1 = \varnothing \\ 3 \text{ if } U \in I(U', \mathcal{U}_2) \text{ and } \mathrm{int}(U) \cap G_1 \neq \varnothing. \end{cases}$$

Suppose that for an integer $n \geq 2$ the position $P(U')$ of U' is defined to be an ordered $(n-1)$-tuple of integers, for each $U' \in \mathcal{U}_n$. Let $U \in \mathcal{U}_{n+1}$ and $U' \in \mathcal{U}_n$ be such that $U \subset U'$. Define the position $P(U)$ of U to be the ordered n-tuple $(k_1, k_2, \dots, k_n)$, where $(k_1, k_2, \dots, k_{n-1}) = P(U')$ and

$$k_n = \begin{cases} 0 \text{ if } U \in B(U', \mathcal{U}_{n+1}) \text{ and } \operatorname{int}(U) \cap G_n \neq \varnothing \\ 1 \text{ if } U \in B(U', \mathcal{U}_{n+1}) \text{ and } \operatorname{int}(U) \cap G_n = \varnothing \\ 2 \text{ it } U \in I(U', \mathcal{U}_{n+1}) \text{ and } \operatorname{int}(U) \cap G_n = \varnothing \\ 3 \text{ if } U \in I(U', \mathcal{U}_{n+1}) \text{ and } \operatorname{int}(U) \cap G_n \neq \varnothing. \end{cases}$$

By slight abuse of notation we shall also write $(k_1, \dots, k_n) = (P(U'), k_n)$.

Definition 14. *If $U, V \in \mathcal{U}_n$ and $P(U) = P(V)$, we shall say that U and V are similarly placed (or that U and V are in similar position).*

We shall define weight functions w_i on $\mathcal{U}_i$, $i \geq 2$ satisfying the following conditions (1_i), (2_i) and (3_i), where for all $U_1, U_2, U_3 \in \mathcal{U}_i$ we let $U_1', U_2', U_3' \in \mathcal{U}_{i-1}$ be such that $U_k \subset U_k'$ for $k = 1, 2, 3$. Let $P(U_k) = (x_{k,1}, \dots, x_{k,i-1})$ for each $k \in \{1, 2, 3\}$. These conditions are designed to ensure that the most efficient paths lie along the graphs G_i.

- (1_i) Let $k, \ell \in \{1, 2, 3\}$. If $x_{k,m} = x_{\ell,m}$ for each $m \leq j$ where $1 \leq j < i-1$ and $x_{k,m}, x_{\ell,m} \in \{0, 1\}$ for $m \in \{j+1, \dots, i-1\}$, then $w_i(U_1) < w_i(U_2) + w_i(U_3)$.
- (2_i) If $U_k \in B(U_k', \mathcal{U}_i)$ for $k = 1, 2$, then $w_i(U_1) \leq w_i(U_2)$ if and only if $w_{i-1}(U_1') < w_{i-1}(U_2')$ or $w_{i-1}(U_1') = w_{i-1}(U_2')$ and $x_{1,i-1} \leq x_{2,i-1}$.
- (3_i) Suppose that $x_{k,m} \in \{0, 1\}$ for $1 \leq m \leq i-1$ and $k = 1, 2, 3$. Then $w_i(U_1) < w_i(U_2) + w_i(U_3)$.

Let $\delta_2 = \frac{1}{4}$. For $U \in \mathcal{U}_2$ let

$$w_2(U) = \begin{cases} 2^{-1} \text{ if } P(U) = 0 \\ 2^{-1} + \delta_2 / m_2^2 \text{ if } P(U) = 1 \\ \delta_2 / m_2^{2k} \text{ if } P(U) = k \in \{2, 3\}. \end{cases}$$

It is easy to see that conditions (1_2), (2_2) and (3_2) are satisfied and two elements U and V of $\mathcal{U}_2$ which are similarly placed have the same 2^{nd}-weight w_2, that is, $w_2(U) = w_2(V)$.

Suppose that the numbers $\delta_i > 0$ and $w_i(U)$ are defined for each integer i with $2 \leq i \leq n$ and each $U \in \mathcal{U}_i$. Suppose that every two elements of $\mathcal{U}_i$ which are similarly placed have the same i^{th}-weight w_i and conditions (1_i), (2_i) and (3_i) are satisfied.

Consider a constant δ_{n+1} such that $0 < \delta_{n+1} < \frac{1}{4} \cdot \min\big(\{w_n(U) : U \in \mathcal{U}_n\} \cup \{|w_n(U) - w_n(V)| : U, V \in \mathcal{U}_n \text{ are in different positions}\}\big)$.

For each $U \in \mathcal{U}_{n+1}$ let $U' \in \mathcal{U}_n$ be such that $U \subset U'$ and let

$$w_{n+1}(U) = \begin{cases} w_n(U')/2 \text{ if } P(U) = (P(U'), 0) \\ w_n(U')/2 + \delta_{n+1}/m_{n+1}^2 \text{ if } P(U) = (P(U'), 1) \\ \delta_{n+1}/m_{n+1}^{2k} \text{ if } P(U) = (P(U'), k) \text{ for } k \in \{2, 3\}. \end{cases}$$

It follows that the elements of $\mathcal{U}_{n+1}$ in similar position have the same $(n+1)^{\text{st}}$-weight. Choose δ_{n+1} sufficiently small so that conditions (1_{n+1}), (2_{n+1}) and (3_{n+1}) are satisfied.

Definition 15. *If K is a non-empty connected subset of X such that $K = \bigcup \mathcal{V}$ for some subcollection $\mathcal{V}$ of $\mathcal{U}_i$, then we shall say that K has i^{th}-weight and we set $w_i(K) = \sum_{V \in \mathcal{V}} w_i(V)$.*

For $p, q \in X$ let $E_i(p, q) = \min w_i(K)$, where the minimum is taken over all K which are subsets of X with i^{th}-weight and with $p, q \in K$. Then $E_i(p, q)$ is the i^{th} approximation of the intended distance from p to q.

For $p, q \in X$ we show $E(p, q) = \lim_{i \to \infty} E_i(p, q)$ exists. Suppose that K is a set which has i^{th}-weight and contains p and q with $w_i(K) = E_i(p, q)$. Then K is the union of a chain $\mathcal{C}$ in $\mathcal{U}_i$. Also, p and q lie in the opposite end-links of $\mathcal{C}$. There is a subset K' of K containing p and q and having $(i+1)^{\text{st}}$-weight such that $w_{i+1}(K') \leq w_i(K) + \delta_{i+1}/m_{i+1}$. It follows as in [3] that $\{E_i(p, q)\}_{i=1}^{\infty}$ is a Cauchy sequence of positive numbers. Thus, $E(p, q)$ is well-defined.

The argument given in [3, p. 547] can be employed to easily prove that E is an equivalent metric for the space (X, d).

4 Auxiliary lemmas

To prove convexity of the metric E defined in Section 3 we shall need the following auxiliary results.

Definition 16. *Let $U, V \in \mathcal{U}_n$ with $P(U) = (x_1, x_2, \ldots, x_{n-1})$ and $P(V) = (y_1, y_2, \ldots, y_{n-1})$. We shall say that U and V have essentially different positions when there is $i \in \{1, 2, \ldots, n-1\}$ such that $x_i \neq y_i$ and $\{x_i, y_i\} \neq \{0, 1\}$. We shall also say that U and V have essentially the same position when they do not have essentially different positions.*

Observe that if $U_1, U_2, U_3 \in B(\mathcal{U}_{n-1}, \mathcal{U}_n)$ have essentially different positions, and $U_k \subset U'_k \in \mathcal{U}_{n-1}$ for $k = 1, 2, 3$, and $w_n(U_1) \geq w_n(U_2) + w_n(U_3)$, then $w_{n-1}(U'_1) \geq w_{n-1}(U'_2) + w_{n-1}(U'_3)$.

For the sake of the following Lemma 1, we let the position of each $U \in \mathcal{U}_1$ be $\varnothing$ (the empty sequence).

Lemma 1. *Let $n \geq 3$ and let $\mathcal{C} = \{C_1, \ldots, C_m\} \subset B(\mathcal{U}_{n-1}, \mathcal{U}_n)$ be a weak chain such that*

$$w_n(C_i) \geq w_n(C_{i+1}) + w_n(C_{i+2}) \text{ for each } i \in \{1, 2, \ldots, m-2\}. \quad (*)$$

Then $m \leq 6$.

Proof. Suppose that $n \geq 3$ and let $\mathcal{C} = \{C_1, \ldots, C_m\} \subset B(\mathcal{U}_{n-1}, \mathcal{U}_n)$ be a weak chain that satisfies $(*)$. It is clear that no three elements of $\mathcal{C}$ have essentially the same position because otherwise (1_n) or (3_n) would be violated. Let $\mathcal{V}'_{n-1} = \{V'_1, \ldots, V'_k\} \subset \mathcal{U}_{n-1}$ be the irreducible weak chain in $\mathcal{U}_{n-1}$ that covers $\bigcup \mathcal{C}$. By $(*)$ and (1_n), each V'_j contains at most 2 links of $\mathcal{C}$. Moreover, there do not exist in $\mathcal{V}'_{n-1}$ three elements in essentially the same position.

Let $\mathcal{V}''_{n-2} = \{V''_1, \ldots, V''_l\} \subset \mathcal{U}_{n-2}$ be the irreducible cover of $\bigcup \mathcal{C}$. Then $\mathcal{V}''_{n-2}$ also covers $\bigcup \mathcal{V}'_{n-1}$. Suppose first that $V'_j \in I(V''_i, \mathcal{U}_{n-1})$. By (d) and (1_n), $\bigcup \mathcal{C} \subset V''_i$. Among all the links of the weak chain $\mathcal{C}$ there are at most 2 links of $\mathcal{C}$ contained in elements $V'_j \in B(V''_i, \mathcal{U}_{n-1})$, at most 2 links of $\mathcal{C}$ contained in elements $V'_j \in I(V''_i, \mathcal{U}_{n-1})$ with $P(V'_j) = (P(V''_i), 2)$ and at most 2 links of $\mathcal{C}$ contained in elements $V'_j \in I(V''_i, \mathcal{U}_{n-1})$ with $P(V'_j) = (P(V''_i), 3)$. So $m \leq 6$ in this case.

We may suppose, therefore, that $\mathcal{V}'_{n-1} \subset B(\mathcal{U}_{n-2}, \mathcal{U}_{n-1})$. Let $\mathcal{V}'''_{n-3} = \{V'''_1, \ldots, V'''_s\} \subset \mathcal{U}_{n-3}$ be the irreducible cover of $\bigcup \mathcal{C}$. Suppose first that $V''_j \in I(V'''_i, \mathcal{U}_{n-2})$. By (d) and (1_n), $\bigcup \mathcal{C} \subset V'''_i$. Using (1_n) similarly there are at most 2 links of $\mathcal{C}$ contained in elements $V''_j \in B(V'''_i, \mathcal{U}_{n-2})$, at most 2 links of $\mathcal{C}$ contained in elements $V''_j \in I(V'''_i, \mathcal{U}_{n-2})$ with $P(V''_j) = (P(V'''_i), 2)$ and at most 2 links of $\mathcal{C}$ contained in elements $V''_j \in I(V'''_i, \mathcal{U}_{n-2})$ with $P(V''_j) = (P(V'''_i), 3)$. Thus, as before, $m \leq 6$.

We may suppose, therefore, that $\mathcal{V}''_{n-2} \subset B(\mathcal{U}_{n-3}, \mathcal{U}_{n-2})$. We continue this process and if we find the smallest positive integer p and a unique element $V_i^{(n-p)} \in \mathcal{U}_{n-p}$ such that $\bigcup \mathcal{C} \subset V_i^{(n-p)}$ then, similarly, as before, $m \leq 6$. The final possibility is when the set $\bigcup \mathcal{C}$ is contained in more than one element of $\mathcal{U}_1$. It means that any two links of $\mathcal{C}$ have essentially the same position and therefore by (3_n) we get $m \leq 2$. □

Lemma 2. *If $\mathcal{C}$ is a chain in $\mathcal{U}_{n+1}$ from p to q such that $w_{n+1}(\bigcup \mathcal{C}) = E_{n+1}(p, q)$ and $U \in \mathcal{U}_n$, then $\{C \in \mathcal{C} : C \subset U\}$ is a chain and there are at most two links $C \in \mathcal{C}$ which belong to $B(U, \mathcal{U}_{n+1})$.*

Proof. Observe that $\bigcup I(U, \mathcal{U}_{n+1})$ is connected and meets each member of $B(U, \mathcal{U}_{n+1})$. Also, $w_{n+1}(V) > w_{n+1}(\bigcup I(U, \mathcal{U}_{n+1}))$ for each $U \in \mathcal{U}_n$ and each $V \in B(U, \mathcal{U}_{n+1})$. Hence, if $\bigcup \mathcal{C} \cap U$ was not connected, then at least two members of $B(U, \mathcal{U}_{n+1}) \cap \mathcal{C}$ would be separated by an element of $B(U, \mathcal{U}_{n+1})$ in $\mathcal{C}$. Then it would be easy to find a chain $\mathcal{C}'$ in $\mathcal{U}_{n+1}$ from p to q such that $w_{n+1}(\bigcup \mathcal{C}') < w_{n+1}(\bigcup \mathcal{C})$. □

Lemma 3. *Let p and q be distinct points of X and $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a chain in $\mathcal{U}_{n+1}$ of smallest $(n+1)^{\text{st}}$-weight such that $p \in C_1$ and $q \in C_m$. Let $U, V \in \mathcal{U}_n$ be such that $C_1 \subset U$ and $C_m \subset V$. Let j'_1 be the maximal index such that $C_{j'_1} \subset U$, and j'_2 be the minimal index such that $C_{j'_2} \subset V$. If $j'_1 + 7 < k < j'_2 - 7$ and $W \in \mathcal{U}_n$ is such that $C_k \subset W$, then there are exactly two members of $\mathcal{C}$ which are in $B(W, \mathcal{U}_{n+1})$.*

Proof. Suppose that the number of links $C \in \mathcal{C}$ which are in $B(W, \mathcal{U}_{n+1})$ is different from

2. By Lemma 2, it follows that $\mathcal{C} \cap B(W,\mathcal{U}_{n+1}) = \{C_k\}$. There are four cases to consider concerning the short chain $\{C_{k-7}, \ldots, C_k, \ldots, C_{k+7}\}$.

Case 1. Suppose there exist integers r and s and $W', W'' \in \mathcal{U}_n$ such that $k-7 < r < k < s < k+7$, $C_{r-1} \cup C_r \subset W'$ and $C_s \cup C_{s+1} \subset W''$. Then each of W' and W'' contains at least two links of $B(\mathcal{U}_n,\mathcal{U}_{n+1}) \cap \mathcal{C}$. Furthermore, $W' \cap W'' \neq \varnothing$, by (d). Let $r' = \min\{i : C_i \subset W'\}$ and $s' = \max\{i : C_i \subset W''\}$. Let $r'' = \max\{i : C_i \subset W'\}$ and $s'' = \min\{i : C_i \subset W''\}$. Also, let $C' \in \mathcal{U}_{n+1}(W')$ and $C'' \in \mathcal{U}_{n+1}(W'')$ be such that $G_n \cap C' \cap C'' \neq \varnothing$, $G_n \cap \mathrm{int}(C') \neq \varnothing$ and $G_n \cap \mathrm{int}(C'') \neq \varnothing$. Note that $j_1' < r' < s' < j_2'$. Let $\mathcal{C}_1$ be a chain in $I(W',\mathcal{U}_{n+1})$ whose first link meets $C_{r'}$ and whose last link meets C'. Also, let $\mathcal{C}_2$ be a chain in $I(W'',\mathcal{U}_{n+1})$ whose first link meets C'' and whose last link meets $C_{s'}$. Then $\mathcal{C}' = \{C_1, \ldots, C_{r'}\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_{s'}, \ldots, C_m\}$ is a chain in $\mathcal{U}_{n+1}$ from p to q such that $w_{n+1}(\bigcup \mathcal{C}') < w_{n+1}(\bigcup \mathcal{C})$, since $w_{n+1}(C_k) > w_{n+1}(\bigcup I(W',\mathcal{U}_{n+1})) + w_{n+1}(\bigcup I(W'',\mathcal{U}_{n+1}))$, $w_{n+1}(C_{r''}) \geq w_{n+1}(C')$ and $w_{n+1}(C_{s''}) \geq w_{n+1}(C'')$.

Case 2. Now, suppose that each element of $\mathcal{U}_n$ which contains at least one member from $\{C_{k-7}, \ldots, C_{k+7}\}$ contains exactly one member of $\mathcal{C}$. By Lemma 1, there exist integers r and s such that $k-7 \leq r \leq k-3 < k+3 \leq s \leq k+7$, $w_{n+1}(C_r) < w_{n+1}(C_{r+1}) + w_{n+1}(C_{r+2})$ and $w_{n+1}(C_s) < w_{n+1}(C_{s-1}) + w_{n+1}(C_{s-2})$. Let $W', W'' \in \mathcal{U}_n$ be such that $C_r \subset W'$ and $C_s \subset W''$. By (d), $W' \cap W'' \neq \varnothing$. Let $C' \in \mathcal{U}_{n+1}(W')$ and $C'' \in \mathcal{U}_{n+1}(W'')$ be such that $G_n \cap C' \cap C'' \neq \varnothing$, $G_n \cap \mathrm{int}(C') \neq \varnothing$ and $G_n \cap \mathrm{int}(C'') \neq \varnothing$. Let $\mathcal{C}_1$ be a chain in $I(W',\mathcal{U}_{n+1})$ whose first link meets C_r and whose last link meets C' and let $\mathcal{C}_2$ be a chain in $I(W'',\mathcal{U}_{n+1})$ whose first link meets C'' and whose last link meets C_s. Hence, as above, for the chain $\mathcal{C}' = \{C_1, \ldots, C_r,\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_s, \ldots, C_m\}$ from p to q we have $w_{n+1}(\bigcup \mathcal{C}') < w_{n+1}(\bigcup \mathcal{C})$. This is because

$$w_{n+1}(C') \leq w_{n+1}(C_r) \leq w_{n+1}(C_{r+1}) + w_{n+1}(C_{r+2}),$$

$$w_{n+1}(C'') \leq w_{n+1}(C_s) \leq w_{n+1}(C_{s-1}) + w_{n+1}(C_{s-2}),$$

and $w_{n+1}(C_k) > w_{n+1}(\bigcup I(W',\mathcal{U}_{n+1})) + w_{n+1}(\bigcup I(W'',\mathcal{U}_{n+1}))$ as $C_k \in B(\mathcal{U}_n,\mathcal{U}_{n+1})$.

Case 3. There exist an integer r and $W' \in \mathcal{U}_n$ such that $k-7 < r < k$, $C_{r-1} \cup C_r \subset W'$ and each element of $\mathcal{U}_n$ which contains at least one of $\{C_k, \ldots, C_{k+7}\}$ contains exactly one member of $\mathcal{C}$.

Case 4. There exist an integer s and $W'' \in \mathcal{U}_n$ such that $k < s < k+7$, $C_s \cup C_{s+1} \subset W''$ and each element of $\mathcal{U}_n$ which contains at least one of $\{C_{k-7}, \ldots, C_k\}$ contains exactly one member of $\mathcal{C}$.

Cases 3 and 4 are similar to the extreme ones considered above. □

Definition 17. *Let* $\mathcal{C}_i = \{U_0, \ldots, U_s\}$ *be a chain in* $\mathcal{U}_i$ *and let* $\mathcal{C}_{i+j} = \{V_0, \ldots, V_r\}$ *be a chain in* $\mathcal{U}_{i+j}$*, where* i *and* j *are positive integers. We shall say that* $\mathcal{C}_{i+j}$ *runs straight through* $\mathcal{C}_i$ *provided* $\mathcal{C}_{i+j} - \mathcal{U}_{i+j}(\bigcup \mathcal{C}_i) = \{V_0, V_r\}$*, and there is an arc* $A \subset G_{i+j-1} \cap \bigcup \mathcal{C}_i$ *such that*

$A \cap (U_1 \cup \cdots \cup U_{s-1}) \subset G_i$, *st*$(A, \mathcal{U}_{i+j}) = \bigcup \mathcal{C}_{i+j}$ *and* $A \cap U_k$ *is an arc with exactly two points in bd*(U_k) *for each* $k = 1, \ldots, s-1$.

Lemma 4. *Let* n *be a positive integer and let* $\mathcal{C}_{n+1} = \{C_1, \ldots, C_m\}$ *be a chain in* $\mathcal{U}_{n+1}$ *such that* $\bigcup \mathcal{C}_{n+1}$ *has smallest* $(n+1)^{\text{st}}$*-weight among all subsets of* X *which have* $(n+1)^{\text{st}}$*-weight and contain* $C_1 \cup C_m$*. Let* $U, V \in \mathcal{U}_n$ *be such that* $C_1 \subset U$ *and* $C_m \subset V$*. Let* j_1 *be maximal such that* $C_{j_1} \subset \text{st}(U, \mathcal{U}_n)$ *and let* j_2 *be minimal such that* $C_{j_2} \subset \text{st}(V, \mathcal{U}_n)$*. Suppose that* $j_1 < j_2$*. Then there exists a chain* $\mathcal{C}_n$ *in* $\mathcal{U}_n$ *such that* $\{C_{j_1}, C_{j_1+1}, \ldots, C_{j_2}\}$ *runs straight through* $\mathcal{C}_n$*. Also,* $\bigcup \mathcal{C}_n$ *has the smallest* n^{th}*-weight among all subsets of* X *which contain* $C_{j_1+1} \cup C_{j_2-1}$ *and have* n^{th}*-weight.*

Proof. Clearly, $w_{n+1}(\bigcup \mathcal{C}_{n+1})$ is smallest among all chains in $\mathcal{U}_{n+1}$ which contain the links $\{C_1, \ldots, C_{j_1}, C_{j_2}, \ldots, C_m\}$ of $\mathcal{C}$. If j'_1 is maximal such that $C_{j'_1} \subset U$ and j'_2 is minimal such that $C_{j'_2} \subset V$, then by condition (d) and the choice of j_1 and j_2, we obtain $j'_1 + 7 < j_1 < j_2 < j'_2 - 7$. Therefore by Lemma 3, each $W \in \mathcal{U}_n$ with $C_j \subset W$ for $j_1 < j < j_2$ contains exactly two links of the chain $\mathcal{C}_{n+1}$ that belong to $B(\mathcal{U}_n, \mathcal{U}_{n+1})$. An argument similar to the one in Case 1 of Lemma 3, implies that $\mathcal{C}_n = \{U_1, \ldots, U_k\}$ must be a chain (not just a weak chain) which is naturally ordered by the order on $\mathcal{C}_{n+1}$ (by Lemma 2). Since $\bigcup \mathcal{C}_{n+1}$ has the smallest $(n+1)^{\text{st}}$-weight among all subsets of X which have the $(n+1)^{\text{st}}$-weight and contain $C_1 \cup C_m$, it follows that $\{C_{j_1}, \ldots, C_{j_2}\}$ runs straight through $\mathcal{C}_n$.

Note that $\bigcup \mathcal{C}_n$ is of the smallest n^{th}-weight among all sets of n^{th}-weight which contain C_{j_1+1} and C_{j_2-1}. Indeed, assume that there is a chain $\mathcal{C}'_n = \{U_1, U'_1, \ldots, U'_l, U_k\}$ in $\mathcal{U}_n$ such that $w_n(\bigcup \mathcal{C}'_n) < w_n(\bigcup \mathcal{C}_n)$. Then there exists a chain $\mathcal{C}'_{n+1}$ in $\mathcal{U}_{n+1}$ with the end links C_{j_1+1} and C_{j_2-1} such that $\bigcup \mathcal{C}'_{n+1} \subset \bigcup \mathcal{C}'_n$ and each link of $\mathcal{C}'_n$ contains exactly 2 links of $\mathcal{C}'_{n+1}$ that belong to $B(\mathcal{U}_n, \mathcal{U}_{n+1})$. But then,

$$w_n\left(\bigcup \mathcal{C}'_n\right) < w_{n+1}\left(\bigcup \mathcal{C}'_{n+1}\right) < w_n\left(\bigcup \mathcal{C}_n\right) < w_{n+1}\left(C_{j_1+1} \cup C_{j_1+2} \cup \ldots C_{j_2-1}\right),$$

where the second inequality is true by the choice of δ_{n+1}. This contradicts the fact that $\bigcup \mathcal{C}_{n+1}$ has the smallest $(n+1)^{\text{st}}$-weight among all sets of $(n+1)^{\text{st}}$-weight and the proof is complete. □

Lemma 5. *Let* i *and* j *be positive integers and let* $\mathcal{C}_{i+j} = \{C_1, \ldots, C_m\}$ *be a chain in* $\mathcal{U}_{i+j}$ *such that* $\bigcup \mathcal{C}_{i+j}$ *has the smallest* $(i+j)^{\text{th}}$*-weight among all sets with* $(i+j)^{\text{th}}$*-weight which contain* $C_1 \cup C_m$*. Let* $U, V \in \mathcal{U}_i$ *be such that* $C_1 \subset U$ *and* $C_m \subset V$*. Let* j_1 *be maximal such that* $C_{j_1} \subset \text{st}^2(U, \mathcal{U}_i)$*, and let* j_2 *be minimal such that* $C_{j_2} \subset \text{st}^2(V, \mathcal{U}_i)$*. Suppose that* $j_1 < j_2$*. Then* $\{C_{j_1}, \ldots, C_{j_2}\}$ *runs straight through a chain* $\mathcal{C}_i$ *in* $\mathcal{U}_i$ *which is of the smallest* i^{th}*-weight among all subsets of* X *which have* i^{th}*-weight and contain the first and last links of* $\mathcal{C}_i$*.*

Proof. Let $U', V' \in \mathcal{U}_{i+j-1}$ be such that $C_1 \subset U'$ and $C_2 \subset V'$. Let j'_1 be maximal such that $C_{j'_1} \subset \text{st}(U', \mathcal{U}_{i+j-1})$ and let j'_2 be minimal such that $C_{j'_2} \subset \text{st}(V', \mathcal{U}_{i+j-1})$. By Lemma 4, there exists a chain $\mathcal{C}_{i+j-1} = \{C'_0, \dots, C'_{k_1}\}$ in $\mathcal{U}_{i+j-1}$ such that $\{C_{j'_1}, \dots, C_{j'_2}\}$ runs straight through $\mathcal{C}_{i+j-1}$, where $\bigcup \mathcal{C}_{i+j-1}$ is of the smallest $(i+j-1)^{\text{st}}$-weight among all subsets of X which have the $(i+j-1)^{\text{st}}$-weight and contain $C_{j'_1+1} \cup C_{j'_2-1}$.

If $j = 1$, then $\mathcal{C}_{i+j-1}$ contains the required chain. If $j > 1$, let $U'', V'' \in \mathcal{U}_{i+j-2}$ be such that $C'_0 \subset U''$ and $C'_{k_1} \subset V''$ and let $\mathcal{C}_{i+j-2} = \{C''_0, \dots, C''_{k_2}\}$ be a chain in $\mathcal{U}_{i+j-2}$ given by Lemma 4 applied to $\mathcal{C}_{i+j-1}$.

One may continue inductively through j steps to get a chain $\mathcal{C}_i$ in $\mathcal{U}_i$ whose first link belongs to $\text{st}^3(U, \mathcal{U}_i)$ and whose last link belongs to $\text{st}^3(V, \mathcal{U}_i)$ such that the subchain $\{C_{j_1}, \dots, C_{j_2}\}$ of $\mathcal{C}_{i+j}$ runs straight through $\mathcal{C}_i$. □

5 Proof of convexity of E and questions

Completion of proof of Theorem 7.

Proof of convexity of the metric E. We shall show that the metric E on X that was constructed above is convex. It is enough to prove that for every $x, y \in X$ there exists an E-segment from x to y which is contained in the set $\{x, y\} \cup \bigcup_{n=1}^{\infty} G_n$.

Let $x \neq y \in X$. We may suppose that $y \notin \text{st}^7(x, \mathcal{U}_1)$ (otherwise it is obvious how to use a similar reasoning starting with some partition $\mathcal{U}_n$). For each $i > 1$ let $\mathcal{C}_i = \{C_{i,1}, \dots, C_{i,p_i}\}$ be a chain in $\mathcal{U}_i$ of the smallest weight among all chains in $\mathcal{U}_i$ which contain x and y. Let j_i^1 be maximal such that $C_{i,j_i^1} \subset \text{st}^3(x, \mathcal{U}_1)$, and let k_i^1 be minimal such that $C_{i,k_i^1} \subset \text{st}^3(y, \mathcal{U}_1)$. Then $j_i^1 < k_i^1$.

By Lemma 5, there exists a chain $\mathcal{E}_i^1$ in $\mathcal{U}_1$ which is of the smallest 1^{st}-weight among all chains of $\mathcal{U}_1$ which contain the first and last links of $\mathcal{E}_i^1$ and such that the chain $\{C_{i,j_i^1}, \dots, C_{i,k_i^1}\}$ runs straight through $\mathcal{E}_i^1$. Since $\mathcal{U}_1$ is finite, there exists a chain $\mathcal{D}_1$ which occurs infinitely often in $\{\mathcal{E}_i^1\}_{i=1}^{\infty}$.

Let $\{\mathcal{C}_{i_l}\}_{l=1}^{\infty}$ be a subsequence of $\{\mathcal{C}_i\}_{i=1}^{\infty}$ such that $\mathcal{E}_{i_l}^1 = \mathcal{D}_1$ for each l. Let $H_1 \subset \bigcup \mathcal{D}_1 \cap G_1$ be the unique arc such that $H_1 \subset \bigcup \mathcal{C}_{i_l}$ for each l.

Replacing $\mathcal{U}_1$ by $\mathcal{U}_2$, there is a chain $\{C_{i_l, j_{i_l}^2}, \dots, C_{i_l, k_{i_l}^2}\}$ in $\mathcal{C}_{i_l}$ for $i_l \geq 3$ with exactly one link in $\text{st}^3(x, \mathcal{U}_2)$ and one in $\text{st}^3(y, \mathcal{U}_2)$, and a chain $\mathcal{E}_{i_l}^2$ in $\mathcal{U}_2$ which is of the smallest 2^{nd}-weight among all chains of $\mathcal{U}_2$ which contain the first and last links of $\mathcal{E}_{i_l}^2$ and such that the chain $\{C_{i_l, j_{i_l}^2}, \dots, C_{i_l, k_{i_l}^2}\}$ runs straight through $\mathcal{E}_{i_l}^2$. Since $\mathcal{U}_2$ is finite, there is a chain $\mathcal{D}_2$ and a subsequence $\{\mathcal{C}_{i_{l_m}}\}_{m=1}^{\infty}$ of $\{\mathcal{C}_{i_l}\}_{l=1}^{\infty}$ such that $\mathcal{D}_2 = \mathcal{E}_{i_{l_m}}^2$ for each m. Let $H_2 \subset \bigcup \mathcal{D}_2 \cap G_2$ be the unique arc such that $H_2 \subset \bigcup \mathcal{C}_{i_{l_m}}$ for each m. Then $H_1 \subset H_2$.

By induction we construct arcs $H_i \subset G_i$ such that $\{x, y\} \cup \bigcup_{i=1}^{\infty} H_i$ is an E-line segment from x to y. □

We conclude the paper with listing several problems concerning generalizations of our construction.

Problem 1. If X is a connected and locally arc-connected (separable) metric space which admits locally finite covers by small open and connected sets, does X admit a convex metric? In other words, is property S in our theorem necessary?

Problem 2. Can the main theorem of the paper be extended to the case of spaces admitting infinite partitions? What if the space has also the disjoint arcs property?

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