CONTINUOUS LINEAR EXTENSION OF FUNCTIONS

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Abstract. Let (X, d) be a complete metric space. We prove that there is a continuous, linear, regular extension operator from the space C_b^* of all partial, continuous, real-valued, bounded functions with closed, bounded domains in X to the space $C^*(X)$ of all continuous, bounded, real-valued functions on X with the topology of uniform convergence on compact sets. This is a variant of a result of Kunzi and Shapiro for continuous functions with compact, variable domains.

1. Introduction

There is a long history of improvements to the Tietze-Urysohn extension theorem. Dugundji [4] proved that if A is a closed subset of a metric space then there is a continuous, linear, regular extension operator from $C(A)$, the space of continuous real-valued functions on A with the topology of pointwise convergence to the space $C(X)$. His operator is also continuous with respect to the topology of uniform convergence on $C(A)$ and $C(X)$.

Kuratowski [7] first considered the space of all continuous partial functions whose domains are closed subsets of a metric space X. The question of existence of operators extending partial functions with variable domains then arose naturally. A non-linear extension operator for partial functions with compact domains was constructed by Stepanova [11].

Kunzi and Shapiro [6] improved the theorems of Dugundji and of Stepanova for functions with compact variable domains as follows:

Let (X, d) be a metric space and

$$C_{vc} = \{f\colon A \to \mathbb{R} \mid A \subset X \text{ is compact and } f \text{ is continuous }\}.$$

Then C_{vc} is a metric space where the distance between two functions f and g is given by the Hausdorff distance between their graphs which are closed, bounded subsets of $X \times \mathbb{R}$. Let $C^*(X)$ denote the set of continuous, bounded, real-valued functions on X.

Theorem 1.1 ([6]). *Suppose that X is a metrizable space and that the set $C^*(X)$ is endowed with the topology of uniform convergence. Then there exists a continuous operator $\Phi\colon C_{vc} \to C^*(X)$ with the following properties:*

1) $\Phi(f)|_{\mathrm{dom} f} = f$ *for every* $f \in C_{vc}$*;*
2) Φ *is regular i.e.*

$$\|\Phi(f)\| = \|f\| = \max\{|f(x)| \mid x \in \mathrm{dom} f\}$$

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for every $f \in C_{vc}$ *and* $\Phi(\mathbf{1}_A) = \mathbf{1}_X$ *for every compact subset* A *of* X*;*

3) *For every compact* $A \subset X$ *the restriction* $\Phi|_{C(A)}$ *is a linear operator i.e.* $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$ *for* $\alpha, \beta \in \mathbb{R}$ *and* $f, g \in C(A)$*.*

It is not proved that the Kunzi-Shapiro operator preserves uniformly continuous functions if X is not compact.

Let

$$C^*_{ub} = \{f\colon A \to \mathbb{R} \mid f \text{ is uniformly continuous and bounded,} \\ A \subset X \text{ is closed and bounded } \}.$$

As in the case of C_{vc} define distance between two functions in C^*_{ub} to be the Hausdorff distance between their graphs. Denote by $C^*_u(X)$ the family of uniformly continuous, bounded, real-valued functions on X. For bounded space X the following result is known:

Theorem 1.2 ([3])**.** *Let* X *be a bounded metric space. There exists a regular, positive homogeneous extension operator* $u\colon C^*_{ub} \to C^*_u(X)$ *which is continuous with respect to the topology of uniform convergence on* $C^*_u(X)$*.*

Note that the extension operator constructed in [3] is not additive and therefore is not linear. The Kunzi-Shapiro theorem's proof seems to depend essentially on the compactness of domains of partial functions. Of course all of the partial functions considered by Kunzi and Shapiro are uniformly continuous.

It is known that the Hausdorff metric convergence of graphs of continuous functions with common domain implies pointwise convergence as well as uniform convergence on compact sets but does not imply the uniform convergence of these functions. However, if the limit function is uniformly continuous then this last implication is true (see [2], [9]). In the current paper we prove a variant of the result of Kunzi and Shapiro for the case of bounded continuous functions defined on all closed, bounded subsets of a complete metric space. We construct an extension operator which is linear, regular and continuous with respect to the topology of uniform convergence on compact sets on $C^*(X)$. In short we weaken the compactness condition on partial domains in the Kunzi-Shapiro result but in return our operator is continuous with respect to the topology of uniform convergence on compact sets on $C^*(X)$ rather than the topology of uniform convergence.

Note that there is wide literature devoted to extensions of functions with special properties (for instance see [12] for a counterpart of the Kunzi-Shapiro theorem for pseudometrics).

2. The main result

Let (X, d) be a metric space and $\exp_b(X)$ the space of non-empty, closed and bounded subsets of X with Hausdorff metric H. Let $d \times H$ be the metric on the product space $X \times \exp_b(X)$ given by the formula

$$(d \times H)[(x, A), (y, B)] = d(x, y) + H(A, B)$$

for every $x, y \in X$ and $A, B \in \exp_b(X)$. Let π_1 and π_2 stand for the projection maps in the product space $X \times \exp_b(X)$.

For $A \in \exp_b(X)$ let $C^*(A)$ denote the family of continuous, bounded, real-valued functions on A. Let

$$C^*_b = \bigcup\{C^*(A) \mid A \in \exp_b(X)\}.$$

We identify each $f \in C_b^*$ with its graph

$$\Gamma_f = \{(x, f(x)) \mid \; x \in \mathrm{dom} f\}$$

which is a bounded and closed subset of $X \times \mathbb{R}$ (here $\mathrm{dom} f$ denotes the domain of f). Let $\tilde{d}$ be the metric on $X \times \mathbb{R}$ defined by

$$\tilde{d}\,((x,t),(x',t')) = d(x,x') + |t - t'|$$

for $(x,t),(x',t') \in X \times \mathbb{R}$. Let $\tilde{H}$ be the Hausdorff metric on $\exp_b(X \times \mathbb{R})$ induced by $\tilde{d}$. For $f, g \in C_b^*$ let distance from f to g be given by $\tilde{H}(\Gamma_f, \Gamma_g)$.

We say $e : C_b^* \to C^*(X)$ is an *extension operator* if for every $f \in C_b^*$ we have $e(f)|_{\mathrm{dom} f} = f$.

We say e is *regular* if

$$\|e(f)\| = \sup\big\{|e(f)(x)| \mid x \in X\big\} = \sup\big\{|f(x)| \mid x \in \mathrm{dom} f\big\} = \|f\|$$

for each $f \in C_b^*$ and $e(\mathbf{1}_A) = e(\mathbf{1}_X)$ for every $A \in \exp_b(X)$ where $\mathbf{1}_A$ is the constant map on A with value equal to 1.

Finally, e is *linear* if $e(af + bg) = ae(f) + be(g)$ for all $f, g \in C_b^*$ with $\mathrm{dom} f = \mathrm{dom} g$ and $a, b \in \mathbb{R}$. The following theorem is the main result of this note.

Theorem 2.1. *Let (X,d) be a complete metric space. There exists a regular, linear extension operator $e : C_b^* \to C^*(X)$. This operator is continuous with respect to the topology of uniform convergence on compact sets on $C^*(X)$.*

Proof. Let $K = \bigcup\big\{A \times \{A\} \mid A \in \exp_b(X)\big\}$ with metric $d \times H$. Then K is closed in $X \times \exp_b(X)$.

Let $\mathcal{U}$ be an open, locally finite cover of $(X \times \exp_b(X)) \setminus K$ such that $U \in \mathcal{U}$ implies $\mathrm{diam}(U) < \frac{1}{2}(d \times H)(z, U)$ for each $z \in K$.

Let $m : Y \to X \times \exp_b(X)$ be an open, exact, Milyutin map with compact fibers of a 0-dimensional metric space Y with a continuous family of fiberwise probability measures on Y

$$\{\mu_{(x,A)}\}_{(x,A) \in X \times \exp_b(X)}$$

(see [1, Theorems 2.1, 2.2]) where the support of $\mu_{(x,A)}$ is $m^{-1}(x,A)$ for each $(x,A) \in X \times \exp_b(X)$.

For $U \in \mathcal{U}$ let

$$W_U = \big\{(a,A) \in K \mid \text{ there exists } (x,A) \in U \text{ with } (d \times H)\big((x,A),(a,A)\big) < 2(d \times H)\big(A \times \{A\},(x,A)\big)\big\}.$$

Then W_U is open in K. For let $(a,A) \in W_U$ and let $\big\{(a_i, A_i)\big\}$ be a sequence in K converging to (a,A). Let $(x,A) \in U$ be such that

$$(d \times H)\big((x,A),(a,A)\big) < 2(d \times H)\big(A \times \{A\},(x,A)\big).$$

Since $\lim_i A_i = A$ and U is open we get $(x, A_i) \in U$ for large i. Now,

$$\lim_i (d \times H)\big(A_i \times \{A_i\},(x,A_i)\big) = (d \times H)\big(A \times \{A\},(x,A)\big).$$

Hence, for large i,

$$(d \times H)\big((x,A_i),(a_i,A_i)\big) < 2(d \times H)\big(A_i \times \{A_i\},(x,A_i)\big),$$

so $(a_i, A_i) \in W_U$ and W_U is open.

Note, that if $\{U_i\}$ is a sequence in $\mathcal{U}$ and $(a, A) \in K \cap \liminf U_i$ then

$$\lim_i W_{U_i} = \big\{(a, A)\big\}. \tag{1}$$

Let $\mathcal{V}$ be a clopen, pairwise disjoint cover of $Y \setminus m^{-1}(K)$ such that $\mathcal{V}$ refines $m^{-1}(\mathcal{U})$. For $V \in \mathcal{V}$ let $U_V \in \mathcal{U}$ with $V \subset m^{-1}(U_V)$. Define a set-valued function $F : Y \to X \times \exp_b(X)$ as follows. Let $z \in Y$. If $z \notin m^{-1}(K)$ there exists a unique $V \in \mathcal{V}$ with $z \in V$. Let

$$F(z) = \begin{cases} m(z) \text{ if } m(z) \in K; \\ \{(a, A) \in W_{U_V}\} \text{ if } \pi_2(m(z)) = A \text{ and } z \in V \in \mathcal{V}. \end{cases}$$

To show that F is lower semicontinuous let $\{z_i\}$ converge to z in Y. Suppose first that $z \notin m^{-1}(K)$. Let V be the unique element of $\mathcal{V}$ such that $z \in V$. Since V is open we may suppose $z_i \in V$ for each i. Let $m(z) = (x, A)$ and $m(z_i) = (x_i, A_i)$. Since m is continuous $\lim m(z_i) = m(z)$. If $(a, A) \in F(z)$ then $(a, A) \in W_{U_V}$ and $a \in A$. Since $\lim A_i = A$ there exists $a_i \in A_i$ such that $\lim a_i = a$. For large i $(a_i, A_i) \in W_{U_V}$ since W_{U_V} is open. So $(a_i, A_i) \in F(z_i)$ for large i and $\lim(a_i, A_i) = (a, A)$ as required.

Now assume that $z \in m^{-1}(K)$ and let $m(z) = (x, A)$ where $x \in A$. Then $F(z) = (x, A)$. By (1) $(x, A) \in \lim F(z_i)$. So F is lower semicontinuous.

Define a set-valued function $\overline{F} : Y \to X \times \exp_b(X)$ by setting $\overline{F}(z) = \overline{F(z)}$ for $z \in Y$. Then, $\overline{F}$ is also lower semicontinuous and has closed point values in the complete metric space $(X \times \exp_b(X), d \times H)$ (see [5, page 298, 4.5.23(c)]). By the 0-dimensional Michael selection theorem [8] $\overline{F}$ has a continuous selection $\varphi : Y \to X \times \exp_b(X)$. Note that $\varphi\big(m^{-1}(x, A)\big) = \{(x, A)\}$ if $(x, A) \in K$.

Define an operator $e : C_b^* \to C^*(X)$ by setting for $g \in C_b^*$ and $x \in X$

$$e(g)(x) = \int\limits_{m^{-1}(x, \mathrm{dom} g)} g(\pi_1(\varphi(z)))d\mu_{(x, \mathrm{dom} g)}.$$

The fact that $e(g)$ is a continuous, bounded function on X follows from continuity and boundedness of the function $g \circ \pi_1 \circ \varphi$ on $m^{-1}(X \times \{\mathrm{dom} g\})$ and continuity of the measures. Indeed, for a sequence $\{x_i\}$ from X converging to some $x_0 \in X$ we obtain

$$\begin{aligned} e(g)(x_i) = \int\limits_{m^{-1}(x_i, \mathrm{dom} g)} g(\pi_1(\varphi(z)))d\mu_{(x_i, \mathrm{dom} g)} = \\ \int\limits_{m^{-1}(X \times \{\mathrm{dom} g\})} g(\pi_1(\varphi(z)))d\mu_{(x_i, \mathrm{dom} g)} \xrightarrow[i\to\infty]{} \\ \int\limits_{m^{-1}(X \times \{\mathrm{dom} g\})} g(\pi_1(\varphi(z)))d\mu_{(x_0, \mathrm{dom} g)} = \\ \int\limits_{m^{-1}(x_0, \mathrm{dom} g)} g(\pi_1(\varphi(z)))d\mu_{(x_0, \mathrm{dom} g)} = e(g)(x_0). \end{aligned}$$

If $x \in \mathrm{dom}g$ then $\varphi(m^{-1}(x, \mathrm{dom}g)) = \{(x, \mathrm{dom}g)\}$ and so

$$e(g)(x) = \int\limits_{m^{-1}(x,\mathrm{dom}g)} g(x) d\mu_{(x,\mathrm{dom}g)} = g(x).$$

Therefore, e is an extension operator. Since integration is a linear operation e is linear.

From the definition of e we see that $\|e(g)\| \leq \|g\|$ for every $g \in C^*_b$ because

$$\mu_{(x,\mathrm{dom}g)}(m^{-1}(x, \mathrm{dom}g)) = \mu_{(x,\mathrm{dom}g)}(Y) = 1$$

for every $x \in X$. And since e is the extension of g we get $\|e(g)\| = \|g\|$. Also, it is clear that e maps $\mathbf{1}_A$ to $\mathbf{1}_X$ for every $A \in \exp_b(X)$. Therefore, e is regular.

Let us show that the map e is continuous with respect to the topology of uniform convergence on compact sets on $C^*(X)$. In fact, it is enough to show that if a sequence $\{g_n\}$ converges to g in C^*_b and if $\{x_n\}$ is a sequence in X converging to $x \in X$ then the sequence $\{e(g_n)(x_n)\}$ converges to $e(g)(x)$. This is the condition of continuous convergence of functions which for metric spaces is equivalent to the uniform convergence on compact sets (see [10, page 109]). Suppose that $\{g_n\}$ converges to g in C^*_b and let $\{x_n\}$ be a sequence in X converging to $x \in X$.

Case 1. Let $(x_n, \mathrm{dom}g_n) = (x, \mathrm{dom}g)$ for every $n \in \mathbb{N}$. Since all the functions g_n and g have the same domain and their graphs converge in the Hausdorff metric, we conclude that $\{g_n\}$ converges pointwise to g on $\mathrm{dom}g$ (see [9]). Then the sequence $\{g_n \circ \pi_1 \circ \varphi\}$ and the limit function $g \circ \pi_1 \circ \varphi$ satisfy the hypothesis of Lebesgue's dominated convergence theorem and we obtain $e(g_n)(x_n) = e(g_n)(x) \to e(g)(x)$.

Case 2. Now suppose that $(x_n, \mathrm{dom}g_n) \neq (x, \mathrm{dom}g)$ for every $n \in \mathbb{N}$. Let

$$Y' = m^{-1}(x, \mathrm{dom}g) \cup \bigcup_{n=1}^{\infty} m^{-1}(x_n, \mathrm{dom}g_n).$$

Define a function $h \colon Y' \to \mathbb{R}$ as follows:

$$h(z) = \begin{cases} g(\pi_1(\varphi(z))) \text{ if } z \in m^{-1}(x, \mathrm{dom}g); \\ g_n(\pi_1(\varphi(z))) \text{ if } z \in m^{-1}(x_n, \mathrm{dom}g_n). \end{cases}$$

Since the sequence $\{\Gamma_{g_n}\}$ converges to Γ_g in the Hausdorff metric one can show that if a sequence $\{a_n\}$ in X converges to $a \in \mathrm{dom}g$ and $a_n \in \mathrm{dom}g_n$ for each n then $g_n(a_n)$ converges to $g(a)$. Using this condition we conclude that h is a continuous map on Y'. Since the measures $\{\mu_{(x_n,\mathrm{dom}g_n)}\}_{n=1}^{\infty}$ converge to $\mu_{(x,\mathrm{dom}g)}$ we obtain

$$e(g_n)(x_n) = \int\limits_{m^{-1}(x_n,\mathrm{dom}g_n)} g_n(\pi_1(\varphi(z))) d\mu_{(x_n,\mathrm{dom}g_n)} =$$

$$\int\limits_{Y'} h(z) d\mu_{(x_n,\mathrm{dom}g_n)} \xrightarrow[n\to\infty]{} \int\limits_{Y'} h(z) d\mu_{(x,\mathrm{dom}g)} =$$

$$\int\limits_{m^{-1}(x,\mathrm{dom}g)} g(\pi_1(\varphi(z))) d\mu_{(x,\mathrm{dom}g)} = e(g)(x).$$

To prove the convergence for any sequence $\{(x_n, g_n)\}$ we will have to pass to subsequences to which Case 1 or Case 2 apply. Therefore, $e(g_n)(x_n) \to e(g)(x)$.

$\square$

Remark 2.2. If $A \in \exp_b(X)$ and a sequence of partial functions $\{g_n\}$ in $C^*(A)$ converges uniformly to $g \in C^*(A)$ on A then the sequence of extensions $e(g_n)$ converges uniformly to $e(g)$ on X. Indeed, since e is linear and preserves norms, we obtain

$$\|e(g) - e(g_n)\| = \|e(g - g_n)\| = \|g - g_n\| \xrightarrow[n\to\infty]{} 0.$$

Remark 2.3. If $g \in C_b^*$ is uniformly continuous its extension $e(g)$ belongs to $C^*(X)$. One cannot conclude that $e(g)$ is uniformly continuous on X because the continuous family of measures $\{\mu_{(x,A)}\}_{(x,A)\in X\times\exp_b(X)}$ cannot, in general, be taken to be uniformly continuous.

Therefore, the following questions arise naturally:

Question 2.4. Suppose that (X, d) is bounded. Does there exist a linear, regular extension operator u from C_b^* to $C^*(X)$ such that $\Gamma_{u(g_n)}$ converges to $\Gamma_{u(g)}$ in the Hausdorff metric whenever $\{g_n\}$ converges to $\{g\}$ in C_b^*? If g is uniformly continuous does $u(g_n)$ converge to $u(g)$ uniformly on X?

Question 2.5. Does there exist a linear, regular extension operator from C_{ub}^* to $C_u^*(X)$ which is continuous with respect to the topology of uniform convergence on $C_u^*(X)$?

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